A Matter of Style and Praxis.
Segre vs. Peano on the Concept of Rigour in Mathematics Education

ERIKA LUCIANO*

1. *Introduction*

The term rigour has always been strictly associated with the figure of Peano: 'master of rigour', 'in all discussions he was always the most rigorous', ... In actual fact, dozens of calls for rigour in the treatment of mathematical notions occur in Peano's writings, with an incidence matching perhaps only that of his appeals for simplicity and clarity of exposition.

If we then extend the discourse to the School of Peano (whatsoever this very controversial term is supposed to mean), the emphasis on mathematical rigour seems to become even more consistent, to the point that it can be considered one of the characteristic features of the style and research program developed by this team in the late nineteenth and early twentieth centuries. The nature and place of rigour in mathematics are considered by Peano from a threefold perspective (scientific, didactic and linguistic), with different nuances and significance according to the stage in question of this mathematician's personal and professional trajectory. However, it is above all from the educational point of view that the interventions of Peano and his School regarding rigour are frequent and detailed, which is why we will focus mainly, almost exclusively on this aspect in this paper. I hope that some preliminary considerations will justify the choice of such a lens of analysis, which is surely atypical and that, frankly speaking, could appear to be really marginal.

First of all, Peano is surely not the only Italian mathematician to have been actively engaged in the fields of mathematical instruction and education. Rather, between the late 19th century and the 1920s, many of the greatest Italian mathematicians showed a particular sensitivity towards methodological issues, sometimes spurred by their own scientific expertise, at other times by interests of a philosophical and epistemological type or by some kind of social commitment.

It was in this cultural setting that the members of the two research Schools blossomed in Turin (those of G. Peano and C. Segre), addressed a wide range of topics including the nature and role of rigour in mathematics education, the criteria for choice of primitive ideas and axioms, the adherence of axiomatic-deductive systems to physical and psychological reality, and the impacts of formal languages, in particular the ways of schematising language, between the opposite poles of natural narrative and the formalism of mathematical logic.

Furthermore, it is the very constitution of the School of Peano that made the philosophy of mathematics education one of the main lines of study cultivated by the team. In fact, differently from all other national and international research equips, the School of Peano was not an academics-based team, but a group consisting mainly of middle and secondary school teachers (37 out of 42 members, with a very peculiar gender balance) who flanked their

* Erika Luciano, Dipartimento di Matematica G. Peano, Università di Torino, via C. Alberto 10 – I 10123 Torino; e-mail: erika.luciano@unito.it.
Abbreviations: CSSUT = Centro Studi per la Storia dell'Università di Torino; DSSP = Deputazione Subalpina di Storia patria; «RdM» = «Rivista di Matematica» (dir. G. Peano, 1891-1906); APVT = Archivio Peano-Vacca, università di Torino.

research activity in logic and foundations with an intense teaching practice at any order and level of instruction, from primary schools up to *licei*. In light of these two justifications, this paper sets out to pinpoint the different ways in which rigour and intuition were delineated by the Peanians in relation to both pedagogic theory and teaching practice and to examine the debates that so-called rational teaching ignited within the context of the Italian mathematical community.

2. *The School of Peano and Italian Mathematical Logic*

In order to properly understand the concept of mathematical rigour in education as envisioned by the School of Peano, we ought to briefly summarize the features of the research program carried out by this group. They are far from being univocally determined[1] however, in general terms, it may be said that the Peanian team created and developed the so-called Italian mathematical logic, i.e. an address of foundational studies characterized by the use of a special formal language (ideography), which first appeared in Peano's writings of 1888-89, and consisted of around ten symbols to denote logical connectives and quantifiers, and around one hundred signs to 'translate' classical mathematics. This systematizing/formalizing attitude led to the affirmation of a very peculiar view of logic, considered by Peanians – even after the rise of Hilbert meta-mathematics – to be a tool in the service of mathematics and mathematics education, useful (indeed indispensable) for rigorizing mathematical thinking and for explicitating or clarifying those hypotheses and deductions that would otherwise be improperly expressed in natural language.

The various chapters of mathematics translated into symbols by Peano and his pupils were brought together in the *Formulario*, an encyclopaedic work drafted in 1891 and published in five editions between 1894 and 1908[2]. Designed specifically as a collective task, for twenty years the *Formulario* represented the magna carta – so to say – of the Italian School of Mathematical Logic. In its complete or final version, the *Formulario* included over 5,000 propositions completely expressed in ideography and a set of extra-mathematical comments (i.e. references to literature and sources, historical annotations) written in *latino sine flexione*. The *Formulario* was a *mother-child* text, i.e. mathematical contents were printed in larger font size, while annotations were printed in a smaller font. This choice of layout well reflected the dual nature of the *Formulario*, which was conceived not only as a 'catalogue of mathematical truths'[3], designed to clarify definitions and theories then current and make them more rigorous, but also as an auxiliary text for teaching and teacher training.

[1] Cf. ERIKA LUCIANO, *Characterizing a Mathematical School. Oral Knowledge and Peano's Formulario*, «Revue d'histoire des Mathematiques», 23, pp. 1-49. Cf. also ERIKA LUCIANO- CLARA SILVIA ROERO, La Scuola di Giuseppe Peano, in CLARA SILVIA ROERO (ed.) *Peano e la sua Scuola fra matematica, logica e interlingua*, Torino, DSSP-CSSUT, 2010, pp. 1-112.

[2] Cf. CLARA SILVIA ROERO, *The "Formulario" between Mathematics and history*, in FULVIA SKOF (ed.) *Giuseppe Peano between Mathematics and Logic*, Milano, Springer, 2011, pp. 83-132, in particular *Tables* 1-5, pp. 108-132.

[3] GIUSEPPE PEANO, *Sul § 2 del Formulario, t. II: Aritmetica*, «RdM», 6, 1898e, p. 85. All the works by Peano are now available in CLARA SILVIA ROERO (ed.), *L'Opera Omnia e i Marginalia di Giuseppe Peano (with English Version*), Torino, Dip. Mat., dvd-rom n. 2, 2008. Peano's papers are quoted according to the classification in this dvd.

3. ***The Peanian educational project***

Since its earliest days, the research project of the School of Peano had clear educational implications and intents[4]. Peano and his protégés were very explicit in affirming that foundational studies constituted a common front between research and teaching and that it was rightly the practical role of foundational studies that justified their importance.
Peano, for example, opened his second work on the principles of geometry by writing:

> Numerous textbooks of Geometry come to light each year in Italy and abroad. […] Now, these new textbooks, in general, do not take into account those observations and special studies carried out on the Foundations of Geometry, which studies, although conducted for purely scientific purposes, can yet make it possible, to a certain extent, to simplify the principles of Geometry, making them more rigorous[5].

The same message was repeated in 1902, when G. Vacca and Peano remarked that they had been driven to elaborate a new deductive theory for Euclidean geometry because the systems that Peano, M. Pieri and A. Padoa had developed (based on point and motion), were fully rigorous but not at all didactic[6]. Lastly, the concrete usefulness of foundational research was recapped in 1910 during one of Peano's last lectures on the foundations of mathematics, held in Turin at the Mathesis association:

> The study of these philosophico-didactic questions is, above all, satisfying to the human mind in its continual search for truth. […] Furthermore, this study is, in essence, of immediate usefulness – on a level with a discovery which would allow us to travel faster, or would lower the price of bread – because a knowledge of these questions, and how to solve them, has the effect of perfecting our teaching, of speeding up the advance of our students, and of giving them the necessary fund of information at the cost of less effort[7].

The edition of the *Formulario*, and in particular that of the chapters of arithmetic, elementary algebra and Euclidean geometry (which constitute almost 70% of the entire work) simultaneously stimulated a closer examination of the defects in the way these disciplines were usually presented in classrooms. It is thus not surprising that Peano, Padoa, Pieri, R.

---

[4] ERIKA LUCIANO, *Aritmetica e Storia nei libri di testo della scuola di Peano*, in LIVIA GIACARDI (ed.), *Da Casati a Gentile: momenti di storia dell'insegnamento secondario della matematica in Italia*, La Spezia, Agorà; Lugano, Lumieres Internationales, 2006, pp. 269-303; EAD., *The Proposals of the School of Peano on the Rational Teaching of Geometry*, in *"Dig Where You Stand" 2. Proceedings of the Fourth International Conference on the History of Mathematics Education*, Lisboa, 2012, UIED, Várzea da Rainha Impressores SA, pp. 31-47.

[5] GIUSEPPE PEANO, *Sui fondamenti della Geometria*, «RdM», 4, 1894c, p. 51: "Numerosi trattati di Geometria veggono la luce ogni anno in Italia e all'estero […]. Ora questi nuovi trattati non sono, in generale, perfezionamenti di quelli già pubblicati, ma vi si ripetono, sotto varie forme, le cose contenute in altri, e non si tien conto di quelle osservazioni e studii speciali fatti sui fondamenti della Geometria,i quali studii, benché fatti con scopo puramente scientifico, pur tuttavia fin d'ora permettono, in certa misura, di semplificare, rendendoli più rigorosi, i principii della Geometria".

[6] APVT: G. Vacca a G. Vailati, 30 settembre - 6 novembre 1902.

[7] GIUSEPPE PEANO, *Sui fondamenti dell'Analisi*, «Bollettino della Mathesis», giugno, 1910a, pp. 31-32: "Lo studio di queste questioni filosofico-didattiche è anzitutto una soddisfazione della mente umana, alla continua ricerca della verità. […] Ma essenzialmente questo studio è di utilità immediata al nostro prossimo, al pari di una scoperta, che ci permetta di correre più veloci, o che abbassi il prezzo del pane. Perché la conoscenza di quelle questioni, e del modo di risolverle, ha per effetto di perfezionare il nostro insegnamento, di far procedere più veloci gli alunni nello studio, e dare a minor prezzo di fatica le cognizioni necessarie".

Bettazzi and C. Burali-Forti published in the last years of the nineteenth century many pamphlets entitled *Note critiche a* … where, in light of foundational research, they unmasked implied admissions, pseudo-definitions and pseudo-proofs, identified the overabundance of primitive ideas and those mathematical truths that did not descend from premises through pure logical deduction[8]. According to the School of Peano, the publication of the *Formulario*, the most perfect model of formal rigour, would therefore offer the solution to a three-fold problem (scientific, linguistic and didactic), 'condemning to the stake'[9] many original research articles and manuals lacking rigour and precision (and thus having no value).

Alongside the joint research program, the School of Peano engaged in collectively establishing and defending a set of instances in the realm of philosophy of mathematics education. Because we cannot enter into further details on all the multiple facets of the Peano School's pedagogical program and commitment, we will limit ourselves to underlining how that of Peanians was an impressive, 360-degree action plan that included: the interplay with various ministers for the reform of mathematics programs (1905, 1912), the publication of no less than 86 school texts of arithmetic and geometry between 1898 and 1915, the organization of courses and lectures on logic and foundations for teachers training, and the collaboration with all the main Italian educational journals of the time[10].

4. *Education to Rigour*

The methodological slogan that better summarizes all of these initiatives can be said to be: 'education to rigour', as opposed to 'education to intuition'[11], which is the manifesto of the 'rival' School of Segre. So, the questions that arise spontaneously are: what is rigour? How is it possible to teach it? How is it possible to educate to it? Or, more precisely, how should that particular form of rigour, usually referred to as mathematical rigour, arise within the teaching of mathematics?

First of all, according to Peano and his followers, mathematical rigour is not an accessory, a superfluous element, something that can be added or imposed *ab extra*. It must come up *ab intra*, both spontaneously and naturally, through the internal needs of the teaching, and encompass and shape the entire structure of teaching in an organic way.

Secondly, rigour historically occurs along with rational mathematics, through a millennial path in which Peano distinguishes two defining moments: Euclidean and modern rigour. Euclidean rigour has its apex in four constructions: the theory of proportions and the method of exhaustion by Eudoxus of Cnidus, the theory of the irrationals by Theaetetus, and Euclidean geometry. Modern rigour, which thanks to foundational research refines and surpasses Euclidean rigour, leads to the definitive establishment of all the necessary

[8] Cf. ALESSANDRO PADOA, *Note critiche agli Elementi di Geometria di Giuseppe Veronese*, Pinerolo, Chiantore-Mascarelli, 1899; CESARE BURALI-FORTI, *Note scientifiche e critiche alle Lezioni di Aritmetica pratica*, Torino, Gallizio, 1897.
[9] GUIDO ASCOLI, *I motivi fondamentali dell'opera di Giuseppe Peano*, in *In memoria di Giuseppe Peano*, Cuneo, Liceo Scientifico Statale, 1955, p. 29.
[10] Cf. FERDINANDO ARZARELLO, *La Scuola di Peano e il dibattito sulla didattica della matematica, in La matematica italiana tra le due guerre mondiali*, Bologna, Pitagora, 1987, pp. 25-41; CLARA SILVIA ROERO, *Giuseppe Peano, geniale matematico, amorevole maestro*, in RENATA ALLIO (ed.), *Maestri dell'Ateneo torinese dal Settecento al Novecento*, Torino, Stamperia artistica nazionale, 2004, pp. 138-144.
[11] Truly speaking, Segre uses 'intuition' as an umbrella term for different kinds of evidence (diagrams, experiments, models, empirical data, etc.).

postulates, even those that have remained tacit for many centuries, and explains their logical value. This is the 'true' rigour, according to Peanians, the absolute rigour which fully satisfies the mathematician and which should be instilled into students.

In the third place, rational teaching is the most suitable form of education for two reasons: because it is the only one suited to guaranteeing the principle of economy of thought and because the natural evolution of all sciences towards the structure of axiomatical-deductive systems (a conviction derived by some papers of E. Goblot[12] and F. Masci[13]) implies the natural evolution of any form of teaching towards the rational approach.

However, education to rigour cannot overlook cognitive issues and material conditions of teaching. This is why, in teaching practice, appeals to intuition flourish, along with the overabundance of implied admissions etc. Peano is not a priori opposed to any sort of attenuation of rigour in the didactic sphere, but he is categorical on some points. Intuition can play a leading role, but only in the stage of choice of primitive ideas and axioms. In this regard, one might think that no room was left for intuition in the pedagogical frame developed by a research School which had constructed hypothetico-deductive systems for geometry and arithmetic, explicitly stating that the ideas of point, space or number were not determined *a priori* and were not associated with any meaning; rather, they were characterised by formal conditions, freely imposed, and were resolved in the set of all the interpretations that they satisfied, subject only to the constraint of consistency. One could argue that Peanians would suggest presenting arithmetic and geometry in classrooms as abstract speculative systems, introducing mathematical objects as "pure creations of the spirit", "artefacts of the mind and truths by definition", the postulates as "simple acts of will"[14], arbitrary in so far as their ordering is determined by the deductive aims set by the author. Nothing could be further from the truth. Unlike G. Halsted or G. Veronese, no member of the School of Peano arrived at denying or marginalizing the didactic value of the physical interpretations of primitive entities and the practical/experimental nature of the postulates. Bettazzi and Levi, indeed, who had a special interest in the psychological foundations of mathematics, remarked on many occasions that the selection of primitive ideas had to be the most appropriate not only from the point of view of logical rigour, but also from that of perception[15], and stated that there was no reasonable way, for example, to avoid presenting geometry in schools as a mathematical physics of the extended bodies, a guise that history, educational traditions and the results of cognitive research necessarily conferred on it[16]. So, Peanians and some bridge figures between the School of Peano and that of Segre (G. Fano, E. Artom) recommended that teachers appeal to any kind of rough and empirical means in order to provoke and bring to life in pupils' minds all sorts of intuitive and experimental knowledge of various mathematical

---

[12] EDMOND GOBLOT, *Essai sur la classification rationnelle des sciences*, Paris, Alcan, 1898.

[13] FILIPPO MASCI, *Sulla natura logica delle conoscenze matematiche. Contribuzione alla teoria della conoscenza*, «Filosofia delle Scuole Italiane», 32, 1885, pp. 3-51, 115-150, 273-293.

[14] MARIO PIERI, *Sur la géométrie envisagée comme un système purement logique*, in *Bibliotèque du Congrès Intern. de Philosophie*, vol. III *Logique et histoire des Sciences*, Paris, Colin, 1901, p. 373.

[15] Cf. RODOLFO BETTAZZI, *Il fanciullo e la matematica*, Torino, Paravia, 1939; BEPPO LEVI, *Esperienza e intuizione in rapporto alla propedeutica matematica. Lettera aperta al Prof. Giovanni Vailati*, «Bollettino di Matematica», VI, 1907, pp. 177-186.

[16] MARIO PIERI, *Sur la géométrie envisagée comme un système purement logique*, in *Bibliotèque du Congrès Intern. de Philosophie*, vol. III *Logique et histoire des Sciences*, Paris, Colin, 1901, p. 377.

objects[17]. A speck of dust, or a pinpoint in a sheet of paper could be usefully exploited to provide an image of a point. Systems of rigid rods arranged in simple structures, or threads attached to a frame, could assist experimental verifications of axioms. The role of intuition was consequently well defined: intuition was the main criterium for choice of primitive ideas and axioms.

If intuition intervenes in the choice of primitive concepts and postulates, rigour shapes everything that follows. Rigour is the main feature in the logical development of any theory. As a consequence, once primitive objects and axioms are introduced - selected according to intuition – the unique aim of mathematics teaching is to train and promote

> the habit of inflexible exactness in reasoning; that is, the sure knowledge of the logical relationships between principles and consequences: in short, the art or capacity of correctly arguing and concluding[18].

So, for example Pieri suggested:

> After the very first lessons the teacher ought to address his disciples thus: Allow me the truth of these primitive propositions and I will accustom you, step by step, by means of successive deductions, to recognize the truth of all the other geometric propositions. The axioms are like the seeds of all geometric truths […]: but the fruits of these do not grow from the seeds if they are not fertilized by reason. In this way Geometry and Algebra, for example, can be constructed; in brief, in what consists the deductive method, which informs all of pure mathematics.[19]

According to Peano, the standards of mathematical rigour requested in teaching practice have to be coherent and commensurate with the intellectual maturity of the students. In particular, G. Vailati, Padoa and Fano proposed dedicating the last year of secondary studies to an overall rethinking of the principles, to make the students understand the high demands of modern rigour and how the refined foundational critique has satisfied them. At lower education levels – they stated – it was useless to propose certain modern constructions. Pupils could actually come to understand them, but they would understand neither their spirit, nor the role of rigour, which should appear to them as a decorative facade superimposed on an existing mathematical building.

An example will suffice to support this last assumption: Padoa presented a new approach to rational numbers in a lecture held at the Congress of Mathesis Association in Padua in 1909 where he proposed to introduce rational number a / b as the class of all the ordered pairs

---

[17] GINO FANO, *Matematica esatta e matematica approssimata*, «Bollettino della Mathesis», III, 1911, p. 106-126; ID., *Sui fondamenti della geometria*, «Rivista di Filosofia», VII, 1915, pp. 374-408; EMILIO ARTOM, *Una lezione alla Scuola Normale sul concetto di piano*, «Bollettino di Matematiche e di Scienze Fisiche e Naturali», XIV, 1, 1912, pp. 1-5; ID., *I fondamenti dell'aritmetica (Una lezione in II Normale)*, «Bollettino di Matematiche e di Scienze Fisiche e Naturali», XIV, 2, 1912, pp. 17-22.

[18] MARIO PIERI, *La geometria elementare istituita sulle nozioni di "punto" e "sfera"*, «Memorie di matematica e di fisica della Società Italiana delle Scienze detta dei XL», 3, 15, 1908, p. 107: "la pratica del ragionare con esattezza; vale a dire la cognizione sicura dei rapporti logici di principio e conseguenza: insomma l'arte o la facoltà di rettamente argomentare e concludere".

[19] *Ibidem*: "La prima volta il Maestro così parli ai discepoli: 'Concedetemi la verità di codeste proposizioni primitive; ed io vi conduco man mano, per via di successive deduzioni, a dover riconoscer la verità di tutte le altre proposizioni geometriche. […]: ma i germi di queste non si svolgono da quelle, se non siano fecondati dal raziocinio. A questo modo s'istituisce, per es., la Geometria e l'Aritmetica; in ciò consiste sommariamente il processo deduttivo, che informa tutta quanta la matematica pura".

of integers proportional to *a* and *b*. This intrinsic theory of fractions was introduced into teaching practice by Padoa himself in Genoa (*Istituto tecnico Vittorio Emanuele*) and the author considered this experience a successful example of transfer into teaching of foundational studies[20]. Far from sharing Padoa's belief, many other Peanians challenged the intrinsic theory of fractions. In particular, according to Artom, F. Palatini, S. Catania and Vacca, teachers with a weak culture in foundational studies easily risked to reduce it to a mere formal game, particularly popular with mediocre pupils and – on the contrary – capable of producing in talented pupils a horror for logical rigour that would be unlikely to be overcome later. As Vacca wrote:

> I ask myself if these objects, which are (it's true!) in univocal and mutual correspondence with the rational numbers, are the same rational numbers that Mankind has known and used for thirty centuries. I believe that the precise and rigorous duty of teachers of mathematical sciences at any school level should consist in teaching what has been given a name for centuries. If one wants, or thinks it is useful, to introduce new objects, he must give them new names, leaving to the old names the sense they have. […] But it would not be a remedy to create an infinite number of new, boring, artificial objects, to introduce useless symmetries or unnecessary rigour. Mathematics is a part, so it seems to me, of the living language and, just as in the living language writers have very little power to modify and create, so also in mathematics, symbols, words, ideas and concepts are transmitted down through the generations, and they constitute a patrimony that should be preserved, and preserved in good condition[21].

So, in this case a very well-founded theory like Padoa's was criticized for its excessive rigour, which altered the historical and linguistic orders through which the various number classes had been introduced.

Rigour must be taught gradually and progressively also in textbooks. It was therefore not important to rigorously define or demonstrate everything. On the contrary, Peano recommended that it was much better to be silent, to suppress, to thin out the information: "simply by suppressing some mathematical definitions found in school texts we gain in rigour and simplicity". Also in this case I give just one example: A. Pensa's *Elementi di Geometria ad uso delle Scuole secondarie inferiori*[22]. This book, published on the recommendation of Burali-Forti by a former student of Peano in the courses of Infinitesimal calculus at the Turin University, assumed as its scientific underpinnings Pieri's paper *La Geometria elementare*

---

[20] ALESSANDRO PADOA, *Introduzione alla teoria delle frazioni*, «Bollettino della Mathesis», 1, 1909, pp. 66-81; ID., *Appendice alla teoria delle frazioni*, «Bollettino della Mathesis», 2, 1910, pp. 4-7; ID., *Dalle frazioni alla libertà d'insegnamento*, «Bollettino di Matematica», 9, 1910, pp. 124-128.

[21] GIOVANNI VACCA, *Sulla teoria dei numeri razionali … e sulla filosofia delle matematiche*, «Bollettino di Matematica», 9, 1910, p. 13: "Io mi domando se questi enti, che sono, è vero, in corrispondenza univoca e reciproca con i numeri razionali, siano proprio i numeri razionali che da trenta secoli gli uomini conoscono e adoperano. Io ritengo infatti che il dovere preciso e rigoroso degli insegnanti di qualunque grado delle scienze matematiche debba consistere nell'insegnare ciò che da secoli ha ricevuto un nome. Se si vuole o si crede utile introdurre nuovi enti, ad essi si devono dare nuovi nomi, lasciando ai nomi antichi il senso che essi hanno. […] Ma non sarebbe un rimedio quello di creare un numero indeterminato di enti nuovi, noiosi, artificiali, per introdurre simmetrie inutili o rigore superfluo. La matematica fa parte, così a me sembra, della lingua viva, e come nella lingua viva gli scrittori hanno un ben piccolo potere di modificare e creare, così anche in matematica i simboli, le parole, le idee, i concetti, sono dalle generazioni passate trasmesse alle successive, e costituiscono un patrimonio che conviene custodire e tenere in buono stato".

[22] ANGELO PENSA, *Elementi di Geometria*, Torino, Petrini-Gallizio, 1912.

istituita sulle nozioni di punto e sfera[23]. In place of the concept of a sphere, however, a very intuitive idea of distance was chosen, i.e. the material example of a cord stretched between two points, or that given by the compass. The treatment was distinguished by its simplicity and at no point was it dry or boring, thanks to the fact that the concern for logic rigour of the overall treatment resided solely in the mind of the author[24]. Physical interpretations were provided for all of the primitive ideas, renouncing those pseudo-definitions that critical research had unmasked as vicious circles, and inserting in the text an evocative set of drawings. Moreover, making the suggestions of Peano his own, Pensa suppressed the majority of proofs, and substituted them with experimental justifications, obtained through superposition. The text might appear as the epitome of intuition; on the contrary, it was extremely appreciated by Peanians. In fact, according to Pieri, Burali-Forti and Peano, this was the best form of transposition of Pieri's system, since its complete scientific justification "was clearly not possible … even in upper-level secondary school"[25].

So, the real meaning of the research into the foundations of mathematics is that it clarifies that the rigour or logical correctness of a proof does not depend on the number or nature of the assumptions or admissions used in it, but rather on the way in which they are used[26]. A proof which uses very numerous assumptions or premises, a proof which uses assumptions quite different from those that in the subsequent systematic treatment will be listed as axioms, does not cease for this reason to be rigorous and it can conveniently contribute to educating and refining the pupil's skills in correctly arguing and concluding. It is therefore not important, in a textbook, to start from a minimal system of independent axioms. What really matters is that every hypothesis or assumption appealed to, be clearly recognized and explicitly formulated, whatever the reasons that led the teacher to take it as a starting point of the reasoning. There is nothing to prevent (in a more advanced course of studies) the same propositions that were assumed as hypotheses being drawn as theorems.

Moreover, education to mathematical rigour is closely linked to education to linguistic rigour, which in turn necessarily entails the teaching of a symbolic language, suitable for replacing natural discourse. This was a very controversial aspect of the rational teaching also within the Peanian group. In fact, as early as 1898 (II edition of the *Formulario*) Peano maintained that logical signs should be introduced in primary and middle schools. In this sense, the translation of Euclid's books V to X into ideography took on a special significance[27]. In turn, many of Peano's collaborators recognised that ideography was a valuable tool

[23] PIERI, *La geometria elementare istituita sulle nozioni di "punto" e "sfera"*, 1908 cit., p. 345-450.
[24] CESARE BURALI-FORTI, *Prefazione* to ANGELO PENSA, *Elementi di Geometria*, Torino, Petrini-Gallizio, 1912, p. III.
[25] BURALI-FORTI, *Prefazione*, 1912 cit., p. IV: "la giustificazione scientifica richiede l'intero sistema del Pieri, sistema che non è possibile sviluppare completamente neanche in una scuola media superiore".
[26] GIOVANNI VAILATI, *G.B. Halsted, Rational Geometry. A textbook for the science of Space, based on Hilbert's foundations, New York, 1906*, «Bollettino di Bibliografia e Storia delle Scienze Matematiche (Loria)», 8, 1905, pp. 74-77.
[27] Cf. GIUSEPPE PEANO, *Les propositions du cinquième livre d'Euclide, réduites en formules*, «Mathesis» (Gand), 10, 1890d, pp. 73-74; ID., *Sommario dei libri VII, VIII, IX di Euclide*, «RdM», 1, 1891d, pp. 10-12; ID., *Sommario del libro X d'Euclide*, «RdM», 2, 1892c, pp. 7-11; ID., *Sul libro V di Euclide*, «Bollettino di Matematica», IV, 1906c, pp. 87-91.

> in virtue of the intellectual powers which [it was] capable of educating and developing, and also for certain of its evocative capacities, which often pointed the way to observations and investigations that would otherwise go unseen[28].

Despite this, teachers like Padoa, Vailati, Burali-Forti, A. Segre, E. Bachi and A. Osimo were perfectly aware of the risks connected to a completely formal treatment of geometry and arithmetic arguments, and they warned their colleagues against an indiscriminate introduction of symbols.

Finally, from the point of view of teachers training, according to Peano, teachers must have a broad and deep culture in logic and foundations. If it is true, in fact, that such questions cannot be treated in school *hic et nunc*

> it is however necessary that the teacher know the solution, or solutions, of them, so that he may know how to select the best, and not repeat just that one which he studied in school; and it is essential that he know the questions that do not have solution, on which he has to be silent. In fact, a person who does not know well the foundations of any part whatever of mathematics will always remain hesitant, with an exaggerated fear of rigour[29].

5. *The Debate on Rigour and Intuition: Peano vs. Segre?*

The Peanian pedagogical program met with many criticisms and gave rise to one of the most famous debates in the history of mathematical education – the debate on rigour and intuition, which involved the international community and, in Italy, opposed the Italian 'logicians' belonging to the School of Peano and the 'geometers', disciples of Segre[30]. At the origin of the clash, which had as its frame the *Rivista di Matematica* edited by Peano, was an article by Segre, entitled *Su alcuni indirizzi nelle investigazioni geometriche. Osservazioni dirette ai miei studenti*[31]. It was actually a long paper, in which Segre set out his views on the method of work in scientific research, on the best ways to train to research young students specializing in geometry and, *en passant*, on the relationship between rigour and intuition. In particular, after observing that "often it will be found convenient to alternate between the synthetic method which appears more penetrating, more enlightening, and the analytical

---

[28] MARIO PIERI, *Della geometria elementare come sistema ipotetico deduttivo: monografia del punto e del moto*, «Memorie della Reale Accademia delle Scienze di Torino», 2, 49, 1899, p. 177: "in virtù degli abiti intellettuali, che i metodi e le dottrine di questa scienza si manifestan capaci di educare e promuovere; ed anche per certa loro facoltà suggestiva, che guida spesso ad osservazioni e ricerche non curate altrimenti".

[29] PEANO, *Sui fondamenti dell'Analisi*, 1910a cit., p. 32: "ma è necessario che l'insegnante conosca la soluzione, o le soluzioni di esse, affinché sappia scegliere la migliore, e non ripetere quella sola che ha studiato in scuola; ed essenzialmente conosca le questioni che non hanno soluzione, e su cui si deve tacere. Chi non conosce bene i fondamenti d'una parte qualunque della matematica, rimane sempre titubante, e con un'esagerata paura del rigore".

[30] Cf. LIVIA GIACARDI, *The Italian School of Algebraic Geometry and Mathematics Teaching in Secondary Schools. Methodological Approaches, Institutional and Publishing Initiatives*, «International Journal for the History of Mathematics Education», 5, I, 2010, pp. 1-19; ID., *Segre's University Courses and the Blossoming of the Italian School of Algebraic Geometry*, in GIANFRANCO CASNATI-ALBERTO CONTE-LETTERIO GATTO-LIVIA GIACARDI-MARINA MARCHISIO-ALESSANDRO VERRA (eds.), *From Classical to Modern Algebraic Geometry. Corrado Segre's Mastership and Legacy*, Basel, Birkhäuser, pp. 3-91.

[31] CORRADO SEGRE, *Su alcuni indirizzi nelle investigazioni geometriche*, «RdM», 1, pp. 42-65, transl. by J.W. Young in 1904, and published with the title *On some tendencies in geometric investigations* in «Bulletin of the American Mathematical Society», 2, 10, pp. 442-468.

which in many cases is more powerful, more general, and more rigorous",[32] Segre stated three considerations: sometimes the creative mathematician can and/or has to sacrifice rigour in the process of discovering new truths, the moment of discovery being more important than the rigorous formulation[33]. According to Segre a mathematician cannot be really content with a result which he has obtained by non-rigorous or incomplete procedures; he will not feel sure of it and satisfied until he has rigorously proved it. But he will not reject a priori the imperfect methods, when he is unable to substitute better ones, since the history of science teaches him about their usefulness[34]. Lack of rigour in the methods should not be confused with errors in reasoning or in the results[35]. In order to avoid falling into error, one needs skill, prudence and practice. For example, after having begun to read a paper by F. Enriques, Segre wrote to G. Castelnuovo:

> I heartily recommend rigour, rigour, rigour […] Ponder carefully over your writings and if you meet with some obstacle, do not overlook this)[36].

These considerations were at the origin of an immediate reply by Peano who categorically stated that the lack of rigour is in no way excusable, nor can a result be considered as achieved until it is rigorously demonstrated. Papers lacking rigour do not advance mathematics one step further.[37] The absolute rigour required in mathematics does not mean that one cannot study a science until all its principles have been analyzed, but the task of mathematics (which, according to Peano, is a 'refined logic'[38]) is to deduce absolutely rigorous consequences from premises chosen by induction.

> Those who state consequences that are not contained in the premises can make poetry, but not mathematics[39].
>
> Absolute rigour, while it is a necessary condition of any scientific work, is still not a sufficient condition. Another condition lies in the hypotheses from which we start. If an author starts from hypotheses contrary to experience, or from hypotheses that are not verifiable by experience, he can deduce (it is true!) some wonderful theory, but such to make exclaim: how wonderful this could have been, if the author had applied his reasoning to practical hypotheses[40]!

Peano's *Remarks* were followed by a *Declaration* of Segre and a *Response* appended by Peano, which both hardened their points of view. Peano's criticisms were in reality mainly addressed to the hyperspace theory dealt with 'geometric' methods, as is evident from another

[32] ID., *Su alcuni indirizzi …*, 1891 cit., p. 52, Engl. transl. 1904, p. 453.
[33] ID., *Su alcuni indirizzi …*, 1891 cit., p. 53, Engl. transl. 1904, p. 454.
[34] *Ibidem*.
[35] *Ibidem*.
[36] C. Segre a G. Castelnuovo, 27.5.1893: "Raccomando poi caldamente il rigore, il rigore, il rigore. Pensi bene a ciò che scrive: e se incontra qualche intoppo non ci passi sopra".
[37] GIUSEPPE PEANO, *Osservazioni del Direttore sull'articolo precedente*, «RdM», 1, 1891, p. 66.
[38] ID., *Osservazioni del Direttore …*, 1891 cit., p. 67.
[39] *Ibidem*: "Chi enuncia delle conseguenze che non sono contenute nelle premesse, potrà fare della poesia, ma non della matematica".
[40] *Ibidem*: "Il rigore assoluto, se è condizione necessaria affinché un lavoro sia scientifico, non è ancora condizione sufficiente. Un'altra condizione sta nelle ipotesi da cui si parte. Se un autore parte da ipotesi contrarie all'esperienza, o da ipotesi non verificabili coll'esperienza, né esse, né le loro conseguenze, potrà, è vero, dedurre una qualche teoria meravigliosa, da far esclamare: quale vantaggio, se l'autore avesse applicato il suo ragionamento ad ipotesi pratiche!".

lively controversy with Veronese in the same months. However, the debate also had significant educational implications. In fact, in his lectures at the Teachers Training School, speaking of the role of rigour in mathematics education, Segre returned to some of the *Remarks addressed to his students*, adding that the postulates must be intuitive and not necessarily independent[41] and that teachers should avoid very obvious postulates (for example, 'the successor to a number is a number'), because "a young person cannot understand the purpose" of a series of such statements[42]. With regard to proofs, he observed that it is not necessary to prove propositions that are intuitively evident nor to define for the young, with a long and boring discourse, the things that he already knows[43]. For educational reasons – Segre added – a teacher can (indeed must) sacrifice rigour, providing students with sketches of proofs in place of, or prior to rigorous proofs. In fact, only these sketches can teach how discoveries of new results are made in mathematics and can show that perfect rigour is reached by progressive approximations[44]. These sketches are useful to give a more concise and easy to remember idea and are always opportune, when they may replace too heavy or difficult demonstrations, which make teaching boring and arid. As a consequence, teachers do not sin in rigour when they mobilize drawings or models of geometric forms in order to 'see' certain properties which cannot be known or obtained by deductive reasoning alone. The mathematician and even more the mathematics teacher must combine honesty with prudence: any appeal or recourse to intuition must be frankly presented as such[45]. Finally, according to Segre it is not appropriate to over-use symbolisms and formalisms, generally difficult for the students, as well as completely useless from the point of view of the overall cultural formation of the majority[46].

Starting from this controversy, extremely local if you will, the debate widened extensively, eventually involving almost all Italian mathematicians dealing with methodological questions. The interventions for and against rigour and intuition are countless. Over the years, however, they lost their character of pedagogical discussion at a theoretical level, moving on to focus on textbooks 'informed to logical rigour'. This is not so surprising as the School of Peano had been engaged above all in the field of school publishing.

The first textbooks, like Burali-Forti and Ramorino's *Aritmetica e Norme per l'insegnamento nelle scuole elementari* where arithmetic was rigorously carried out as a deductive theory (three primitive concepts with concrete meanings and 6 postulates, the principle of induction making its first occurrence as proof-method in a school textbook) were generally appreciated[47]. That was not, however, the case of handbooks like *Aritmetica generale e algebra elementare* by Peano and M. De Franchis' *Geometria elementare ad uso dei Licei e dei Ginnasi superiori*, which partnered the rational approach with the teaching of

[41] CORRADO SEGRE, [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*], 1888-1920, f. 17.
[42] ID., [*Appunti* …], 1888-1920 cit., ff. 19-20: "I giovani spesso non capiscono lo scopo".
[43] ID., [*Appunti* …], 1888-1920 cit., f. 18: "Definire al ragazzo con lungo discorso delle cose che egli crede già di conoscere è annoiarlo"
[44] ID., [*Appunti* …], 1888-1920 cit., f. 24-25.
[45] ID., *Su alcuni indirizzi* …, 1891 cit., p. 55; Quaderno N. 40, f. 25.
[46] Cf. also GUIDO CASTELNUOVO, *I programmi di matematica proposti per il liceo moderno*, «Bollettino Mathesis», IV, 1912, pp. 120-128.
[47] CESARE BURALI-FORTI, ANGELO RAMORINO, *Aritmetica e norme per l'insegnamento nelle scuole elementari*, Torino, Pellarano, 1898.

ideographic formalism, giving a short account of ideography in their first sections and then developing arithmetic and geometry, as done in the *Formulario*[48].

Despite their editorial success, confirmed by their numerous reprintings, many of these textbooks aroused a huge series of criticisms. Generally, their critics thought that the extensive use of the axiomatic-deductive method, matched with the new logical language, would mask the natural paths of mathematical thought, leading to mechanicism, in addition to overlooking the applications of mathematics in the physical and natural sciences. In essence, it was feared that rational teaching would be suitable only for students who were exceptionally brilliant or exceptionally mediocre.

Furthermore, a serious problem existed, related to the fact that these texts required a constant and complex epistemic-cognitive mediation, which could not be simply entrusted to the good intention of teachers but required adequate teachers training. Testimony gathered regarding the experiences in adopting this type of handbooks effectively shows across Italy a wide range of superficial and rough adaptations and simplifications.

This type of debate led to some ruptures in the Italian mathematical community. From 1905 onward, the methodological instruction that supply the Italian *curricula* reflected these tensions, even more so when Ministry inspectors went so far as to prohibit the adoption of rational handbooks.

However, this debate was not confined within the Italian public, but it had an international resonance. It's just the case to remind the report on rigour and intuition commissioned by ICMI (International Commission for Mathematical Instruction) in 1911. On that occasion, in order to compare the methods employed in the various countries with which he is concerned, G. Castelnuovo focused on just one type of school, high schools with humanistic orientation, and just one area of mathematics, geometry. Taking a number of school textbooks as examples, Castelnuovo divided teaching methods into the following kinds: A) the purely logical method, B) methods based on empirical principles and logical development, C) methods consisting in alternating and mixing intuitive and deductive considerations and D) the intuitive-experimental method. Method A) is describe as follows: "All the axioms are stated. Their independence is discussed; development is strictly logical. No appeal is made to intuition. Primitive entities are subject to a unique constraint: that of satisfying axioms"[49]. According to Castelnuovo not a single country had systematically adopted the rational teaching proposed by Peano, D. Hilbert, and G. Halsted, which had been tried out only by random teachers; in Italy the method B was generally preferred. In the discussion that followed Castelnuovo's report, very intelligently Enriques observed:

> attention must be drawn to the difference between intuitive and experimental methods. It is really noteworthy that logicians are often found among the partisans of the experimental method. For example, in our country the late Vailati opposed any call for intuition but inclined towards the search for logical rigour and for the development of concrete geometric experiences. Part of the School of Peano is of the same opinion. In

[48] GIUSEPPE PEANO, *Aritmetica generale e Algebra elementare*, Torino, Paravia, 1902b; MICHELE DE FRANCHIS, *Geometria elementare*, Milano, Sandron, 1909.

[49] ICMI Report, *La rigueur dans l'enseignement mathématique dans les écoles moyennes*, «L'Enseignemt Mathématique», 13, 1911, p. 462.

> Denmark, the text by Bonnesen is a successful combination of authentic logical rigour and the use of experience[50].

After the outcomes of the debates on rigour and intuition and the ascertainment of the difficulties faced by the teachers who had performed experiences of the rational approach, the School of Peano (and also members such as Padoa, V. Cavallaro and A. Natucci who had been fervent supporters of the Peanian pedagogic program) made a serious self-criticism[51]. The School continued to play an active part in teacher training; however, the textbooks that adopted the rational approach disappeared almost completely, with a few exceptions (such as the manuals by G. Marletta, P. Benedetti and C. Rosati)[52].

6. *Conclusions*

Aside from the outputs, more or less successful, of the commitment of the School of Peano in the field of education, there remains the fact that the contribution of these scholars was by no means insignificant for the evolution of Italian teaching. As Guido Ascoli said, in fact,

> the stake to which Peano damned many works that infested our schools was healthy and the Italian school, after overcoming the exaggerations of some fanatic, made a decisive improvement[53].

The pedagogic program of the School of Peano succeeded in imprinting the Italian tradition in the teaching of mathematics and left a legacy at least until the Gentile Reform. In no other country of the world did the foundational tendency had such a large influence on the educational milieu, especially in the teaching of arithmetic.

We said that Peano examined the role and nature of mathematical rigour from a threefold perspective: scientific, didactic and linguistic. From the scientific point of view he did not change his position over time: from 1891 to his death he continued to maintain that rigour is an essential attribute of all mathematical research, that the history of mathematics is the history of the (instantaneous) discovery of a series of truths and that a mathematical result is achieved only when rigorous and complete proof is obtained. Members of his School, especially Vacca and Padoa, shared these beliefs. From the scientific (i.e. mathematical) perspective the truth is that the two points of view, of Segre and of Peano, which could have been complementary, remain clearly separated because of the 1891 controversy. On one hand

---

[50] ICMI Report, *La rigueur dans l'enseignement mathématique dans les écoles moyennes*, 1911 cit., p. 468: "M. Enriques croit qu'il convient d'appeler l'attention de la Commission sur la différence entre la méthode intuitive et la méthode expérimentale. Il est très remarquable que parmi les partisans de la méthode expérimentale se trouvent souvent des logiciens. C'est ainsi que, chez nous, le regretté Vailati était surtout adversaire de l'appel à l'intuition, mais il voulait d'un côté la rigueur logique, de l'autre côté le développement de véritables expériences géométriques. Dans la même situation se trouve peut-être en partie l'école de Peano. En Danemark, le traité de Bonnesen donne aussi un exemple remarquable de liaison entre une véritable rigueur logique, et l'emploi de l'expérience".

[51] ALESSANDRO PADOA, *Matematica intuitiva*, Palermo, Sandron, 1923; VINCENZO CAVALLARO, *Notione de parallelismo in scholas secundario*, «Schola et Vita», 3, 1928, pp. 80-81; ALPINOLO NATUCCI, *Methodologia didactico pro mathematica*, «Schola et Vita», 3, 1928, p. 269.

[52] GIUSEPPE MARLETTA, *Trattato di geometria elementare ad uso delle scuole secondarie*. Catania, Giannotta, 1911; PIERO BENEDETTI, CARLO ROSATI, *Geometria per i ginnasi, licei ed istituti tecnici*. Napoli, Perrella, 1924.

[53] ASCOLI, *I motivi fondamentali dell'opera di Giuseppe Peano*, 1955 cit., p. 29: "il rogo cui dannò tante opere che infestavano le nostre scuole fu salutare […] e la scuola italiana, superate le esagerazioni di qualche fanatico, ne ha avuto un miglioramento decisivo".

there was Peano, who at the time had developed a perfect capacity to express modern axiomatic theories in an adequate and extremely synthetic language, but who however made a very limited use of such language, in order to “freeze” in a deep and elegant way classical theories rather than with the aim of producing new knowledge; on the other hand there was Segre who had perfectly envisioned the creative use of axiomatic abstraction in order to open new fields of research more or less unexplored, but who had a completely inadequate vision of the language and the techniques necessary for a rigorous and complete development of an axiomatic theory.

Mathematics education, both in the sense of ‘philosophy of mathematics education’ and in that of ‘concrete teaching praxis’, is a component that is anything but marginal in an overall and complete reconstruction of the Peano’s school epistemology. In this sense, the analysis of the nature and the role assigned by the Peanians to mathematical rigour in the realm of teaching can provide a useful complement to the works of Avellone-Brigaglia, Borga-Freguglia-Palladino on the controversy between Peano and Veronese (regarding rigour and intuition in hyperspace geometry)[54] or to those of Mancosu on the Peanian style in logic[55].

Moreover, this lens of analysis is particularly suitable in order to show the exchanges and synergies among the few key figures and the numerous humble teachers who belonged to the School of Peano, who either shared the dynamics of joint scientific work typical of the research team and took part in them, with a production that was at times a preliminary stage, and at others a reflection of their teaching practice and experience acquired in real-life classrooms across the country.

From a didactic point of view, Peano integrated and refined his ideas over the years, passing from the principles supported in a somewhat naïve, flickering way in 1891 to more detailed and documented arguments, also in keeping with international influences (of F. Klein, J.W.A. Young, …)[56]. As far as some tenets of education to rigour are concerned, I just invite you to keep in mind how different the Peanian vision of rigour in mathematics education is from that which is attributed to him by Castelnuovo in the ICMI survey.

We can conclude that Peano gradually became more conciliatory (in particular regarding minimality and the independence of axioms selected to underpin the textbooks). However, on the question of an early introduction to formal language he remained steadfast, almost uncompromising and deaf to the feed-back, certainly not all positive, regarding this approach.

From the linguistic point of view, Peano’s convictions remained substantially stable over the years: absolute linguistic rigour is achieved if, and only if, ideographic symbols entirely replace natural language, substituting calculus (i.e. the procedures of logic algorithmic) in the place of ‘classic’ discursive argumentation. Peanians, especially after 1904-05, did not follow the Master in his ‘crusade’ in defense of ideography and indeed, apart from a few exceptions, they did not avail themselves of purely deductive methods of representation.

---

[54] M.ARCO BORGA - PAOLO FREGUGLIA - DARIO PALLADINO (eds.), *I contributi fondazionali della scuola di Peano*, Milano, Angeli, 1985; MAURIZIO AVELLONE - ALDO BRIGAGLIA - CARMELA ZAPPULLA, *I fondamenti della geometria proiettiva in Italia da De Paolis a Pieri*, Palermo, Università, 1998.

[55] Cf. PAOLO MANCOSU, *Mathematical Style*, Stanford Encyclopedia of Philosophy, 2009, 23 p.

[56] Cf. FELIX KLEIN, *Elementarmathematik vom höheren Standpunkte aus*, Berlin, Springer, 1924-27; JACOB W.A. YOUNG, *L’insegnamento delle matematiche nelle scuole elementari e secondarie* [transl. by Dionisio Gambioli], Milano, Sandron, 1924.

REFERENCES

ARTOM EMILIO, *Una lezione alla Scuola Normale sul concetto di piano*, «Bollettino di Matematiche e di Scienze Fisiche e Naturali», XIV, 1, 1912, pp. 1-5.

ARTOM EMILIO, *I fondamenti dell'aritmetica (Una lezione in II Normale)*, «Bollettino di Matematiche e di Scienze Fisiche e Naturali», XIV, 2, 1912, pp. 17-22.

ARZARELLO FERDINANDO, *La Scuola di Peano e il dibattito sulla didattica della matematica, in La matematica italiana tra le due guerre mondiali*, Bologna, Pitagora, 1987, pp. 25-41.

ASCOLI GUIDO, *I motivi fondamentali dell'opera di Giuseppe Peano*, in *In memoria di Giuseppe Peano*, Cuneo, Liceo Scientifico Statale, 1955, pp. 23-30.

AVELLONE MAURIZIO - BRIGAGLIA ALDO - ZAPPULLA CARMELA, *I fondamenti della geometria proiettiva in Italia da De Paolis a Pieri*, Palermo, Università, 1998.

BENEDETTI PIERO - ROSATI CARLO, *Geometria per i ginnasi, licei ed istituti tecnici*. Napoli, Perrella, 1924.

BETTAZZI RODOLFO, *Il fanciullo e la matematica*, Torino, Paravia, 1939.

BORGA MARCO - FREGUGLIA PAOLO - PALLADINO DARIO (eds.), *I contributi fondazionali della scuola di Peano*, Milano, Angeli, 1985.

BURALI-FORTI CESARE, *Note scientifiche e critiche alle Lezioni di Aritmetica pratica*, Torino, Gallizio, 1897.

BURALI-FORTI CESARE - RAMORINO ANGELO, *Aritmetica e norme per l'insegnamento nelle scuole elementari*, Torino, Pellarano, 1898.

CASTELNUOVO GUIDO, *I programmi di matematica proposti per il liceo moderno*, «Bollettino Mathesis», IV, 1912, pp. 120-128.

CAVALLARO VINCENZO, *Notione de parallelismo in scholas secundario*, «Schola et Vita», 3, 1928, pp. 80-81.

DE FRANCHIS MICHELE, *Geometria elementare*, Milano, Sandron, 1909.

FANO GINO, *Matematica esatta e matematica approssimata*, «Bollettino della Mathesis», III, 1911, p. 106-126.

FANO GINO, *Sui fondamenti della geometria*, «Rivista di Filosofia», VII, 1915, pp. 374-408.

GIACARDI LIVIA, *The Italian School of Algebraic Geometry and Mathematics Teaching in Secondary Schools. Methodological Approaches, Institutional and Publishing Initiatives*, «International Journal for the History of Mathematics Education», 5, I, 2010, pp. 1-19.

GIACARDI LIVIA, *Segre's University Courses and the Blossoming of the Italian School of Algebraic Geometry*, in GIANFRANCO CASNATI - ALBERTO CONTE - LETTERIO GATTO - LIVIA GIACARDI - MARINA MARCHISIO - ALESSANDRO VERRA (eds.), *From Classical to Modern Algebraic Geometry. Corrado Segre's Mastership and Legacy*, Basel, Birkhäuser, pp. 3-91.

GOBLOT EDMOND, *Essai sur la classification rationnelle des sciences*, Paris, Alcan, 1898.

ICMI Report, *La rigueur dans l'enseignement mathématique dans les écoles moyennes*, «L'Enseignemt Mathématique», 13, 1911, pp. 461-468.

KLEIN FELIX, *Elementarmathematik vom höheren Standpunkte aus*, Berlin, Springer, 1924-27.

LEVI BEPPO, *Esperienza e intuizione in rapporto alla propedeutica matematica. Lettera aperta al Prof. Giovanni Vailati*, «Bollettino di Matematica», VI, 1907, pp. 177-186.

LUCIANO ERIKA, *Aritmetica e Storia nei libri di testo della scuola di Peano*, in LIVIA GIACARDI (ed.), *Da Casati a Gentile: momenti di storia dell'insegnamento secondario della matematica in Italia*, La Spezia, Agorà; Lugano, Lumieres Internationales, 2006, pp. 269-303.

LUCIANO ERIKA, *The Proposals of the School of Peano on the Rational Teaching of Geometry*, in *"Dig Where You Stand" 2. Proceedings of the Fourth International Conference on the History of Mathematics Education*, 2012, Lisboa, UIED, Várzea da Rainha Impressores SA, pp. 31-47.

LUCIANO ERIKA, *Characterizing a Mathematical School. Oral Knowledge and Peano's Formulario*, «Revue d'histoire des Mathematiques», 23, 2017, pp. 1-49.

LUCIANO ERIKA - ROERO CLARA SILVIA, La Scuola di Giuseppe Peano, in C. S. Roero (ed.) *Peano e la sua Scuola fra matematica, logica e interlingua. Atti del Congresso internazionale di studi Torino, 6-7 ottobre 2008*, Studi e Fonti per la Storia dell'Università di Torino, XVII, Torino, Deputazione Subalpina di Storia Patria, 2010, pp. pp. xi-xviii, 1-212.

MANCOSU PAOLO, *Mathematical Style*, Stanford Encyclopedia of Philosophy, 2009, 23 p.

MARLETTA GIUSEPPE, *Trattato di geometria elementare ad uso delle scuole secondarie*. Catania, Giannotta, 1911.

MASCI FILIPPO, *Sulla natura logica delle conoscenze matematiche. Contribuzione alla teoria della conoscenza*, «Filosofia delle Scuole Italiane», 32, 1885, pp. 3-51, 115-150, 273-293.

NATUCCI ALPINOLO, *Methodologia didactico pro mathematica*, «Schola et Vita», 3, 1928, p. 269.

PADOA ALESSANDRO, *Note critiche agli Elementi di Geometria di Giuseppe Veronese*, Pinerolo, Chiantore-Mascarelli, 1899.

PADOA ALESSANDRO, *Introduzione alla teoria delle frazioni*, «Bollettino della Mathesis», 1, 1909, pp. 66-81.

PADOA ALESSANDRO, *Appendice alla teoria delle frazioni*, «Bollettino della Mathesis», 2, 1910, pp. 4-7.

PADOA ALESSANDRO, *Dalle frazioni alla libertà d'insegnamento*, «Bollettino di Matematica», 9, 1910, pp. 124-128.

PADOA ALESSANDRO, *Matematica intuitiva*, Palermo, Sandron, 1923.

PEANO GIUSEPPE, *Les propositions du cinquième livre d'Euclide, réduites en formules*, «Mathesis» (Gand), 10, 1890d, pp. 73-74.

PEANO GIUSEPPE, *Osservazioni del Direttore sull'articolo precedente*, «RdM», 1, 1891, pp. 66-69.

PEANO GIUSEPPE, *Sommario dei libri VII, VIII, IX di Euclide*, «RdM», 1, 1891d, pp. 10-12

PEANO GIUSEPPE, *Sommario del libro X d'Euclide*, «RdM», 2, 1892c, pp. 7-11.

PEANO GIUSEPPE, *Sui fondamenti della Geometria*, «RdM», 4, 1894c, pp. 51-90.

PEANO GIUSEPPE, *Sul § 2 del Formulario, t. II: Aritmetica*, «RdM», 6, 1898e, pp. 75-89.

PEANO GIUSEPPE, *Sul libro V di Euclide*, «Bollettino di Matematica», IV, 1906c, pp. 87-91.

PEANO GIUSEPPE, *Sui fondamenti dell'Analisi*, «Bollettino della Mathesis», giugno, 1910a, pp. 31-37.

PENSA ANGELO, *Elementi di Geometria*, Torino, Petrini-Gallizio, 1912.

PIERI MARIO, *Della geometria elementare come sistema ipotetico deduttivo: monografia del punto e del moto*, «Memorie della Reale Accademia delle Scienze di Torino», 2, 49, 1899, pp. 173-222.

PIERI MARIO, *Sur la géométrie envisagée comme un système purement logique*, in *Bibliotèque du Congrès Intern. de Philosophie*, vol. III *Logique et histoire des Sciences*, Paris, Colin, 1901, pp. 367-404.

PIERI MARIO, *La geometria elementare istituita sulle nozioni di "punto" e "sfera"*, «Memorie di matematica e di fisica della Società Italiana delle Scienze detta dei XL», 3, 15, 1908, p. 345-450.

ROERO CLARA SILVIA, *Giuseppe Peano, geniale matematico, amorevole maestro*, in R. ALLIO (ed.), *Maestri dell'Ateneo torinese dal Settecento al Novecento*, Torino, Stamperia artistica nazionale, 2004, pp. 138-144.

ROERO CLARA SILVIA (ed.) *L'Opera Omnia e i Marginalia di Giuseppe Peano* (*with English Version*), Torino: Dip. Mat., dvd-rom n. 2, 2008.

ROERO CLARA SILVIA, *The "Formulario" between Mathematics and history*, in F. Skof (ed.) *Giuseppe Peano between Mathematics and Logic*, Milano, Springer, 2010, pp. 83-132.

SEGRE CORRADO, [*Appunti relativi alle lezioni tenute per la Scuola di Magistero*], 1888-1920, in L. Giacardi (ed.) *Corrado Segre e la Scuola italiana di geometria algebrica*, http://www.corradosegre.unito.it

SEGRE CORRADO, *Su alcuni indirizzi nelle investigazioni geometriche*, «RdM», 1, pp. 42-65, transl. by J.W. Young *On some tendencies in geometric investigations*, «Bulletin of the American Mathematical Society», 2, 10, 1904, pp. 442-468.

VACCA GIOVANNI, *Sulla teoria dei numeri razionali … e sulla filosofia delle matematiche*, «Bollettino di Matematica», 9, 1910, pp. 11-15.

VAILATI GIOVANNI, *G.B. Halsted, Rational Geometry. A textbook for the science of Space, based on Hilbert's foundations, New York, 1906*, «Bollettino di Bibliografia e Storia delle Scienze Matematiche (Loria)», 8, 1905, pp. 74-77.

YOUNG JACOB W.A., *L'insegnamento delle matematiche nelle scuole elementari e secondarie,* transl. by D. Gambioli, Milano, Sandron, 1924.