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Bertrand Russell: from Neo-Idealism to Mathematical Logic

ABSTRACT: *This essay is dedicated to the transition from neo-idealism to neo-positivism at Trinity College Cambridge at the beginning of 20th century. In 1903 George Edward Moore's* The Refutation of Idealism *and Bertrand Russell's* The Principles of Mathematics *marked the birth of mathematical logic. Especially Russell's* The Principles of Mathematics *contain the theory of classes and the notion of denoting for the development of neo-positivism, then logical empiricism in the European continent. The objective of the article consists of underline the central position of neo-positivism in the twentieth-century philosophy. Indeed neo-positivism of Trinity College Cambridge has to be linked to the logical empiricism of the Vienna Circle, of the Berlin Circle and of the Lvov-Warsaw School. In that sense, the history of contemporary thought can be interpreted by the duality idealism/anti-idealism.*

KEYWORDS: *mathematical logic, neo-idealism, neo-positivism, notion of denoting, theory of classes.*

1. Introduction

If we scroll through the virtual index of the history of twentieth-century culture, we note that philosophy takes an interest in the issues raised by science and epistemology, to the point of having to question the traditional character of all Western culture. In the context of Trinity College Cambridge, it is worth remembering the publication in the review *Mind*, in 1903, of George Edward Moore's essay "The Refutation of Idealism", but also that of Bertrand Russell's *The Principles of Mathematics*. In 1883 Francis Herbert Bradley had published *The Principles of Logic* and in 1893 *Appearance and Reality*, consolidating the neo-idealism which had established itself in the homeland of empiricism and positivism in 1865 with James Hutchison Stirling's *The Secret of Hegel*. In this connection, following the socio-political and socio-cultural crisis that had arisen in Great Britain around the 1860s, the so-called return to idealism had prevailed. The return to criticism had taken place in Germany at the University of Heidelberg, with the opening speech "Ueber Bedeutung und Aufgabe der Erkenntniss-Theorie" held in 1862 by Eduard Zeller. The return to Hegel had occurred in Great Britain during the reign of Queen Victoria and the premiership of Benjamin Disraeli. Poets and

Filosofia, anno LXVIII, 2023, pp. 165-180 • Mimesis Edizioni, Milano-Udine
Web: https://www.ojs.unito.it/index.php/filosofia • ISSN (print): 0015-1823, ISSN (online): 2704-8195 • DOI: 10.13135/2704-8195/??

men of culture, like Samuel Taylor Coleridge, William Wordsworth, Thomas Carlyle, Ralph Waldo Emerson and John Ruskin, were attracted by the philosophy of idealism (especially that of Schelling and Hegel). Among philosophers we must remember the following: Bernard Bosanquet, Francis Herbert Bradley, Edward Caird, Thomas Hill Green, John Ellis McTaggart and David George Ritchie.

Bertrand Russell was born and grew up in this period. Having lost his parents as a child (his mother Kate Stanley in 1874 and his father John in 1876), he received his first education from his paternal grandparents and then from his brother Frank (for geometry in 1883). In 1890 he enrolled at Trinity College Cambridge, where he began studying mathematics and moral sciences, passing Tripos I in 1893 (in the Mathematical Triposes) and Tripos II in 1894 (in the Moral Sciences Triposes). After completing his university studies, he went to Berlin in 1895, attracted by German social democracy; hence in 1896 he gave a cycle of lectures at the London School of Economics and Political Science on the subject *German Social Democracy*. After lecturing in the United States at Johns Hopkins University and Bryn Mawr College, he returned to Cambridge and was appointed professor at Trinity College in 1899. Here he met and frequented George Edward Moore and Alfred North Whitehead, by whom he was influenced for the transition from the philosophy of idealism to symbolic logic. A first turning point in his formative and cultural process took place in 1900, when he went to Paris to participate in the International Philosophy Congress, during which he met Giuseppe Peano. The Italian mathematician greatly influenced the young Russell for the development of a logical theory expressed with mathematical symbolism. Russell went to Paris with Whitehead, with whom he was to collaborate on writing *Principia Mathematica*. But the meeting with Peano was decisive, as he recalls in his *Autobiography*: "The Congress was a turning point in my intellectual life, because I there met Peano."[1] At the beginning of the 20th century he was to work on developing a new philosophy based on the close relationship between mathematics and logic, as he states in the 1937 "Introduction to the second edition" to *The Principles of Mathematics*: "The fundamental thesis of the following pages, that mathematics and logic are identical, is one which I have never since seen any reason to modify."[2] Hence at the start of the 20th century *The Principles of Mathematics* represented a turning point in European culture.

For Russell's biography and production we must refer to the text of his *Autobiography* in three volumes: the first (published in 1967) concerns the period 1872-1914, the second (published in 1968) concerns the period 1914-1944, the third (published in 1969) concerns the period 1944-1967. For the initial phase of his activity, two important periods can be identified: that of idealism (from 1894 to 1898) and that of symbolic logic (from 1898 to 1913). However, we should not overlook the year of publication (1897) of *An Essay on the Foundations of Geometry*, with which Russell states that he is still under the influence of neo-idealism, in the wake of McTaggart; the essay was appreciated by the Catholic

1 Russell 1995, 147; see also volumes one, two and three, published by G. Allen & Unwin, London, 1967, 1968, 1969.
2 Russell 2010, XXXI; first published in 1903 by Cambridge University Press.

theologian James Ward, but criticized by Whitehead and Moore. The transition from neo-idealism to neo-positivism came in 1899, when he was called to Trinity College to hold a course on Leibniz, replacing McTaggart. From this course in 1900 there emerged *A Critical Exposition of the Philosophy of Leibniz*, with which Russell proceeded on the chronological and thematic segment of the new logic, which he was to expound in *The Principles of Mathematics*.

In the rich and variegated production of European culture, Russell's *The Principles of Mathematics* represents a profound epistemological and philosophical break for the whole of the 20th century. Hence one could resort to the concept of fracture developed by Karl Löwith[3] for nineteenth-century philosophy with the book *Von Hegel zu Nietzsche*, whose subtitle ("Der revolutionäre Bruch im Denken des 19. Jahrhunderts") suggests we should also continue with this interpretation for the history of philosophical and scientific thought in the 20th century. The "cultural revolution" of the 20th century can be found in the year 1903, so that, paraphrasing Löwith, one could rewrite the subtitle of his monograph in these terms: "The Revolution in Twentieth-Century Thought" ("Der revolutionäre Bruch im Denken des 20. Jahrhunderts"). The 20th century, according to Eric Hobsbawm,[4] is the short century marked out by extremes ranging from 1914 (the year of the outbreak of the First World War) to 1991 (the definitive year of the fall of the Soviet Union). From the point of view of the history of culture, the 20th century could be defined as a complex century which, after the end of the First World War, saw the rise of Stalinism as well as Fascism and Nazism; after the Second World War, we should remember the student movement of 1968 and the fall of the Berlin wall in 1989. The 20th century could also be defined as the long century,[5] which, however, cannot be broken up or dissected. Indeed, we should link the 20th century to the 21st century, if we want to reflect on the evolution of philosophy and science, taking into account the authors and currents of thought that have characterized culture in the contemporary age. In this context, *The Principles of Mathematics* highlight the transition from idealism to anti-idealism in the early 20th century, based on the vast and varied production of the British philosopher, who among other things represents the man of culture engaged in civil battles. Among Russell's publications, we can mention the following: *Principles of Social Reconstruction* (1916), *The Practice and Theory of Bolshevism* (1920), *The Problem of China* (1922), *Why I Am Not a Christian* (1927), *Sceptical Essays* (1928), *Marriage and Morals* (1929), *The Conquest of Happiness* (1930), *Education and the Social Order* (1932), and *Power: A New Social Analysis* (1938). Further, after the end of the Second World War, the following are also worth mentioning: *Authority and the Individual* (1949), *Human Society in Ethics and Politics* (1954) and *Common Sense and Nuclear Warfare* (1959). At the end of his long, almost centennial existence, we find *My Philosophical Development* (1959) and *Autobiography* (in three volumes published over the years 1967, 1968 and 1969).

3 Löwith 1941.
4 Hobsbawm 1994.
5 Barraclough 1964.

The philosopher Russell lived from 1872 to 1970. He should be remembered for the turning point he marked in 1903. In effect *The Principles of Mathematics* have taken on historiographical and theoretical relevance in the panorama of philosophy set alongside the new symbolic logic. Published in 1903, they are to be considered as looking forward to *Principia Mathematica*, which Russell produced in three volumes in the years 1910, 1912 and 1913 together with Alfred North Whitehead.[6] 1903 marked the starting year of neo-positivism in the cultural context of Trinity College Cambridge, as opposed to the cultural orientation of Balliol College Oxford, where the return to idealism started and prevailed. Reading Moore's short but significant essay, "The Refutation of Idealism", there is the sign of the turning point within British culture, the relevance of which is to be found in the following years in the context of European culture with the Vienna Circle, the Berlin Circle and the Lvov-Warsaw School. It is no coincidence that Moore wrote his essay to refute idealism and recover the tradition of the philosophy of common sense, dating back to the eighteenth-century Scottish school headed by Thomas Reid. Moore defined himself as a critic and opponent of idealism, but also as a supporter of the philosophy of common sense found in *Principia Ethica*, which also appeared in 1903. In 1925 he published *A Defence of Common Sense* with the aim of supporting a radical realism, with which to underline a sort of scepticism as opposed to idealism and in any case based on the denial of the absolute value of knowledge. It is interesting to underline what Moore writes: "The only *reasonable* alternative to the admission that matter exists *as well as* spirit, is absolute Scepticism – that, as likely as not *nothing* exists at all. All other suppositions – the Agnostic's, that something, at all events, does exist, as much as the Idealist's, that spirit does – are, if we have no reason for believing in matter, as baseless as the grossest superstitions."[7]

To tell the truth, Moore had anticipated his critique of idealism as early as 1899 with the publication in the review *Mind* of the essay "The Nature of Judgment", with which he had affirmed his realism against idealism. Therefore, Moore can rightly be considered the forerunner of the anti-idealism cultivated by Russell, who, however, with *The Principles of Mathematics*, at Trinity College Cambridge, marked the definitive turning point in the direction of the development of neo-positivism and symbolic logic. Russell's work, dedicated to the "principles of mathematics", in 1903 reflects the result of the research carried out in the immediately preceding years (between 1900 and 1902), referred to in the essay "Toward the 'Principles of Mathematics' 1900-1902", inserted in the third volume of the *Collected Papers* edited by the Canadian historian of mathematics Gregory H. Moore.[8] In the 20th century, with *A Critical Exposition of the Philosophy of Leibniz*, Russell had begun to work out his new symbolic logic, which precisely is expounded in a complete form in *The Principles of Mathematics* (1903) and developed in *Principia Mathematica* (1910, 1912 and 1913). In the period between these two works we

6 Whitehead and Russell 1910; 1912; 1913.
7 Moore 1903, 453.
8 Russell 1993, 181-208.

have to notice two essays that are not at all marginal: "On Denoting"[9] (1905) and "Mathematical Logic as Based on the Theory of Types"[10] (1908); we also have to remember the content of "The Philosophy of Logical Atomism"[11] (1918-1919). Continuing to examine Russell's production, one can see a connection with Ludwig Wittgenstein's work "Logisch-philosophische Abhandlung" published in 1921 in *Annalen der Naturphilosophie* and in 1922 in English with the title *Tractatus logico-philosophicus*, with an *Introduction* by Bertrand Russell. The encounter with Wittgenstein suggests not neglecting the 1924 essay "Logical Atomism",[12] which appeared three years after *The Analysis of Mind*[13] as the foundation of empirical knowledge and the logical reconstruction of the act of thinking.

2. From logical algebra to symbolic logic

The text of *The Principles of Mathematics*, as stated by Russell himself in the *Preface* to the first edition of his work, was conceived with two objectives: the first, to underline that "all pure mathematics" is aimed at the treatment of concepts that can be defined as "fundamental logical concepts"; the second, to assert that the fundamental concepts of mathematics are to be considered as "indefinable." It is not by chance that the first part of *The Principles of Mathematics* bears the title "The Indefinables of Mathematics", with the aim of placing at the centre of mathematics the theory of indefinables, which Gottlob Frege and Giuseppe Peano had already dealt with. In 1883 the German mathematician and philosopher distinguished himself with the publication of the first volume on the *Grundgesetze der Arithmetik* (the second volume came out in 1903), contributing to the process of refounding mathematics and logic, with respect to the tradition that was consolidated from Aristotle on. As he had already argued with *Begriffsschrift* in 1879, Frege reiterated the need to recognize the notions of mathematics as "primes", that is, as "indefinables" and to represent them in logical terms; so the first propositions of mathematics are to be taken as indefinable, that is, as intuitively evident. Peano's work (*Arithmetices principia nova methodo exposita*) appeared in 1889, with evident reference to the logical algebra of Leibniz and to the debate that was developing in Europe in the late nineteenth and early twentieth centuries. Peano had relations with Frege, but also with Russell and with Whitehead; he reduced the indefinables to three: "number", "zero" and "subsequent to"; Peano's axioms are in themselves indemonstrable (indefinable) like the postulates of Euclid's plane geometry.

It is no coincidence that in *The Principles of Mathematics* Russell refers to not only Frege but also Peano as regards *The Calculus of Relations*[14] and *Peano's*

9 Russell 1905, 479-93.
10 Russell 1908, 222-262.
11 Russell 1918, 495-527; 1919, 32-63, 190-222, 345-380.
12 Russell 1924, 357-384.
13 Russell 1921.
14 Russell 2010, 23-26.

Symbolic Logic.[15] After Frege, Peano was a fundamental author for Russell, who recognized that he had developed a new symbolism, with which to arrive at the notions of "relationship" and "class." In this way mathematics can be resolved in the new logic of relations, as described in the essay in French "Sur la logique des relations avec des applications à la théorie des series"[16] published in 1901 in *Rivista di Matematica* edited by Peano. Evidently *The Principles of Mathematics* constitute the first complex and multifaceted work with which Russell worked out and expounded the most significant parts of symbolic logic related to pure mathematics. In the index of the book there are seven parts that should be remembered: "The Indefinables of Mathematics" (I), "Number" (II), "Quantity" (III), "Order" (IV) "Infinity and Continuity" (V), "Space and Matter" (VI) and "Motion" (VII). In Russell's work, in addition to the fundamental concepts of symbolic logic, the concepts of "continuity" and "infinity" are dealt with in the context of the theory of classes and the notion of denoting.

The neo-positivism of Trinity College Cambridge confirmed the principles of the new symbolic logic, dating back to the logical algebra of Leibniz. The German philosopher did not neglect the principle of identity based on the formula "A = A", harking back to the dialectical dialogues of Plato (*Theaetetus*, *Sophist* and *Parmenides*) and to Aristotle's work on logic (*Organon*). Leibniz maintained that the principle of identity allows the elaboration of clear and distinct knowledge, to be carried out with the symbols of algebra; in 1666 he completed *Dissertatio de arte combinatoria*. For the sake of completeness, in the context of Leibniz's production, as well as *Dissertatio de arte combinatoria*, we must mention *Specimen quaestionum philosophicarum ex jure collectarum* (written in 1665 to obtain a post in philosophy at the University of Leipzig). The reference to Leibniz's logical algebra suggests a link with Ramon Llull, who in 1274 wrote *Ars compendiosa inveniendi veritatem seu ars magna et maior*, whose content is developed in *Ars demonstrativa* (1275), in *Ars generalis ultima* (1305-1308) and in *Ars brevis* (1308). In Llull's most important work we find the intention of developing a combinatorial art based on general principles, valid for all sciences. Indeed, *ars magna* is conceived with the letters of the alphabet, concentric circles and geometric figures. In Llull's work it is also possible to find a basis of mysticism, typical of the culture of the thirteenth and fourteenth centuries, which however opens up to the demands of the modern age down to Leibniz's logical algebra. The chronological and thematic segment leading from Llull to Leibniz and from Leibniz to Russell does not cover all the history of logic within the context of Western culture. Some modern logicians have believed that Aristotle's logic, based on the traditional syllogism, contained the working out of propositional logic in a nutshell. We owe this interpretation to the Swiss mathematician Leonhard Euler (who in 1768-1772 wrote *Lettres à une Princesse*

15 Russell 2010, 27-33.

16 Russell 1901, 115-148. The essay was first published in 1901 in the *Rivista di Matematica* with the title "Sur la logique des relations avec des applications à la théorie des series" and then in English in 1965 with the title "The logic of relations" in *Logic and Knowledge. Essays 1901-1950*, edited by Robert Charles Marsh, 3-38. London and New York: G. Allen & Unwin.

d'Allemagne. Sur divers sujets de physique et de philosophie), according to whom the first proposition represents a universal set in affirmative terms, the second proposition a universal set in negative terms, the third proposition a particular subject in positive terms, and the fourth proposition a particular subject in negative terms. Hence the Swiss mathematician represented the three propositions of Aristotelian syllogism with the following four figures:

Affirmative universal "All Ps are Qs" ("Every P is a Q")
Negative universal "All Ps are not Qs" ("No P is a Q")
Affirmative particular "Some Ps are Qs" ("There exists a P which is a Q")
Negative particular "Some Ps are not Qs" ("There exists a P that is not a Q").

During the 20th century this hypothesis was taken up by Jan Łukasiewicz with *Aristotle's Syllogistic from the Standpoint of Modern Logic*, published in 1951. According to the Polish logician, it is possible to find in the ancient Stoic philosopher, Zeno of Citium, the character of autonomy of logic with respect to the other two components of philosophy (i.e. physics and ethics). According to the Stoics, logic, divided into dialectics and rhetoric, suggests recognizing the most significant part of logic in dialectics. Specifically, dialectics concern both so-called "signifiers" and so-called "*signifi*és"; for example, the word *horse*, on the one hand, has a phonetic value and, on the other, it refers to the real object. The ancient Stoics became supporters of a logic that did not presuppose any form of ontologism based on essences. Zenonian logic can be classified as a sort of nominalism based simply on the term "utterance" through the voice. Hence another reference could be to Roscelin of Compiègne's nominalism, for which there are no essences, but only "emissions of voice" (*flatus vocis*). Roscelin's nominalism brings us down to William of Ockham, for whom both Aristotle's *Organon* and Porphyry's *Isagoge* suggest deducing that each term is only a "*vox*", that is to say a simple vocal sound, as anticipated by Roscelin. Nominalism is contrasted with ontologism on the basis of a conception of philosophy identified with symbolic logic. This new logic is simply descriptive of things that appear, unlike logic that presupposes the existence of substances, separated from phenomena.

Russell's symbolic logic suggests finding the distinction between the logic of terms and propositional logic; in this case it is necessary to refer to the logic codified by Kant through the judgments set out in *Kritik der reinen Vernunft*. The logic of terms is based on judgments consisting of two variables (the subject and the predicate) and one constant (the verb). The logic of terms represents the presumption of metaphysics that it is able to interpret reality, unlike symbolic logic, which suggests limiting oneself to representing phenomena. The objection of the new logic consists in maintaining the formula "p implies q", instead of the formula "a is b." The propositions "p" and "q" are defined functors, since they are logical constants which limit themselves to representing the relationship existing between the two propositions. Indeed the two propositions "p" and "q" are aimed at the description of the phenomena of reality, without claiming to establish that they are true or false. For example, with respect to the judgment expressed with the formula "all bodies are heavy", to be found in Kant's *Kritik der reinen Vernunft*, the new logic of neo-positivism suggests the formula "if bodies fall, there is a law of gravity." Between the first proposition ("if bodies fall") and the second proposition ("there is a law of gravity") there is a relationship of implication, whereby the former implies the latter, or the latter is inferred from the former. The theory of classes must be related to the theory of relations, which is not by chance expounded in chapter 9 of *The Principles of Mathematics*, specifically entitled "Relations."

According to this approach, the new logic of neo-positivism allows us to overcome the articulation of Aristotelian syllogism focused on the following:

Major premise ("All men are mortal")
Minor premise ("Socrates is a man")
Conclusion ("Socrates is mortal").

The formula suggested by propositional logic would be the following: "Socrates is a man implies Socrates is mortal", or "p implies q." The letter "p" stands for "Socrates is a man", while the letter "q" stands for "Socrates is mortal." The content of both the first and second propositions has an indefinite character, but one which is useful for representation of the phenomena of reality. By asserting that "p implies q", it can be understood that "Socrates is a man implies Socrates is mortal", but also that "Plato is a man implies Plato is mortal." Symbolic logic allows us to overcome the scheme of the Aristotelian syllogism: "All men are mortal, Socrates is a man, Socrates is mortal." The suggestion would be "if a is b and x is a, then x is b"; given that "a" stands for "all men", "b" for "mortals" and "x" for "Socrates." Therefore, a variable meaning is attributed to "x", because it can indicate Socrates, Plato or any other man. The character of symbolic and propositional logic is expounded by Russell in the first part of *The Principles of Mathematics* ("The Indefinables of Mathematics"), and precisely in the second chapter ("Symbolic Logic"), where we read: "the advantage of having before our minds a strictly formal development is that it provides the data for philosophical analysis in a more definite shape than would be otherwise possible."[17] Going on reading the work, it can be seen that Russell does not neglect non-Euclidean geometries as regards projective geometry and descriptive geometry, which had been developed during the second half of the 19th century. In some ways our author takes inspiration from what has already been developed above all by Cantor, Weierstrass and Dedekind, contributing to the process of renewal of twentieth-century culture.

However, Leibniz's name is central in Russell's work, if we wish to recall the assumption of "The Infinitesimal Calculus", set out in section 33 in Part V on "Infinity and Continuity." The theory of infinitesimal calculus constitutes one of the most significant and important parts of the history of humanities and scientific culture, determined at the end of the 17th century with the contribution of the English physicist Newton and the German philosopher Leibniz. The theory of calculus focuses on the analysis of the concepts of infinitesimal and function. These

17 Russell 2010, 18.

two concepts are the basis of modern mathematics, which in the contemporary age was to develop with so-called non-Euclidean geometries, promoted by the Russian Nikolai Ivanovich Lobachevsky, who wrote *New Principles of Geometry* in the years 1835-1838, and the German Bernhard Riemann, the author of the essay "Über die Hypothesen, welche der Geometrie zu Grunde liegen" written in 1854, but published in 1867.

Wishing to focus on the origins of infinitesimal calculus, we must dwell on the contributions made by Newton and Leibniz. Newton closes the happy season of modern science, the origins of which must be traced back to Nicolaus Copernicus, Tycho Brahe, Johannes von Kepler and Galileo Galilei. Newton was not a real mathematician, but a physicist that discovered mathematics thanks to the contribution of his teacher Isaac Barrow, who in 1669 gave him the chair of mathematics at Trinity College Cambridge. In 1676 there began a correspondence between Leibniz and Newton, and a controversy was to develop between them on the paternity of infinitesimal calculus. Newton believed that Leibniz had developed the theory of infinitesimal calculus, taking inspiration from the content of his unpublished essays, but made known in the course of their correspondence, which covered a period of ten years. The reaction from Newton and his supporters came in 1684 with the appearance in *Acta Eruditorum* of Leibniz's essay on "Nova methodus pro maximis et minimis." The long-distance controversy led Newton to accuse Leibniz of plagiarism; in effect, Newton's calculation of fluxions is not identical to Leibniz's theory of functions. Therefore it should be noted that Newton and Leibniz developed infinitesimal calculus simultaneously, but distinctly from each other, with different facets.

For Leibniz's development of infinitesimal (or differential) calculus, the reference texts are both "Nova Methodus pro maximis et minimis" and "De geometria recondita et analysi indivisibilium atque infinitorum", which both appeared in *Acta Eruditorum*, respectively in 1684 and in 1686. It should not be forgotten that Leibniz was known to the public for having written two essays on dynamic physics between 1670 and 1671: "Theoria motus abstracti" presented at the Académie Française in Paris and "Theoria motus concreti" presented at the Royal Society in London. In these two essays, which are part of *Hypothesis physica nova*, the German philosopher dwells on themes concerning the nature of the body, movement, continuity and divisibility of matter. Specifically, he does not neglect the concepts of abstract motion and concrete motion, in relation to the concept of continuity of motion, centred on the concept of "force" (*conatus*), or on the principles of conservation of impulse and energy, already formulated in 1642 by Thomas Hobbes in *De motu, loco et corpore* and by Thomas White in *De mundo*. In the following years Leibniz developed the concept of function, to be applied both to natural numbers (1, 2, 3, …) and to fractions. According to Leibniz, it is possible to assume an infinitely small quantity, to be understood as a differential and to indicate with the letters "dx", where the letter "d" is the abbreviation for difference.

In Newton's case, in addition to his main work dedicated to *Philosophiae naturalis principia mathematica* in 1687, other essays should be mentioned: "De analysi per aequationes numero terminorum infinitas", written in 1669 and published in 1711; "Methodus fluxionum et serierum infinitarum cum eiusdem applicatione ad curvarum geometriam", written in 1671 and published posthumously in 1736. Furthermore, we must not neglect *Tractatus de quadratura curvarum*, written in 1665 and published in 1704. For the English man of science, geometric quantities were not to be conceived as infinitely small entities; infinitesimals were seen as quantities produced by a continuous motion, like, for example, the lines produced by the continuous motion of a point or the surfaces produced by the continuous motion of a line. Therefore, within Newton's production, we find the term "fluxion" in relation to the speed with which a quantity flows from one quantity to another. Newton centred his calculation on the continuous motion of quantity and on the relative speed, supporting the material existence of infinitesimals.

Based on the work (*Arithmetica infinitorum*) by his fellow-countryman John Wallis, Isaac Newton considers quantities as variables in relation to time; therefore, he defines "flowing" quantities and takes into account that a certain speed or "fluxion" corresponds to each instant. The theory of infinitesimal calculus, at the beginning of the 18th century, was criticized by George Berkeley, a representative of the current of thought of British empiricism (headed by David Hume, as well as John Locke). In 1734, Berkeley published an essay with the title "The Analyst" and the emblematic subtitle "A Discourse Addressed to an Infidel Mathematician." In 1721 Berkeley had published the essay "De motu" to criticize Newton's concepts of space, time and motion. With the essay "The Analyst", Berkeley intended to object to the limit of mathematics in claiming to interpret reality; he wanted to support his theory of knowledge in clear controversy with Newton's theses on the existence of an absolute space and an absolute time. Berkeley suggested believing in the existence, not only of relative space and relative time, but also of relative motion, given that the distance of a body changes with respect to the position of the observer or another body. Addressing Newton in critical terms, Berkeley objected that the computation of fluxions could correspond to the computation of derivates.

Berkeley's goal was not so much to argue against mathematics, but to argue that mathematicians often rely on abstract concepts that are meaningless, as in the case of the infinitesimal. Berkeley's objections were already overcome during the second half of the 18th century and in the following two centuries. The theory of infinitesimal calculus was codified especially during the nineteenth and twentieth centuries, to the point of forming the foundation of the new non-Euclidean geometries. Among those who recognized the legitimacy of the infinitesimal calculus in a purely mathematical field, it is necessary to remember the name of the French mathematician Augustin-Louis Cauchy, for whom the infinitesimal should not be understood as an infinitesimal quantity that is completely evanescent, though different from zero. According to Cauchy, the infinitesimal must be conceived as a quantity that tends to cancel itself out, being an infinitely small or an infinitely large. Therefore the concept of infinitesimal must be placed in close relationship with the concept of infinity, infinity being that which has no end both in the direction of the infinitely small and in the direction of the infinitely large. The concepts of limit and continuity are the basis of the theory that Cauchy expounded in his most significant essays: *Le Calcul infinitésimal* (1823) and *Leçons sur les applications du calcul infinitésimal à la géométrie* (1826-1828).

3. Theory of classes and notion of denoting

Reflecting on the content of *The Principles of Mathematics*, we can and must deduce that Russell's contribution at the beginning of the 20th century is central to the entire history of philosophical and scientific thought. The reference is to "The Philosophy of the Infinite", specifically treated in chapter 43 at the conclusion of Part V, not surprisingly entitled "Infinity and Continuity." We know that the concepts of continuous (or continuity) and infinite (or infinity) imply an immediate reference to the keywords already coded in the classical age. The parts to be found within Russell's work, in addition to the theory of classes, concern the notion of types, but also the notion of denoting. In this context, the concepts of contradiction and antinomy, placed in close relationship with the concepts of continuity and infinity, should be noted. Besides, everyone knows that, among the most significant keywords in the history of philosophical and scientific thought, we find those of *continuum* and *infinitum* which in *The Principles of Mathematics* are dealt with in Part V ("Infinity and Continuity"). Within the latter we find chapters 42 and 43, dedicated to "The philosophy of the continuum"[18] and "The philosophy of the infinite."[19] These two chapters are to be set alongside chapter 39, which focuses on "The Infinitesimal Calculus."[20] The keyword *continuum* represents continuity in both space and time; according to Aristotle, the *continuum* constitutes a complex set of problems, expounded in the fifth book of *Physics*,[21] which on a strictly philosophical plane leads to Hegel's *Wissenschaft der Logik*. On the specific plane of mathematics, Russell combines the terms continuous and infinite with that of infinitesimal, with the aim of demonstrating that "The infinitesimal, therefore – so we may conclude – is a very restricted and mathematically very unimportant conception, of which infinity and continuity are alike independent."[22] Therefore he refers to the contributions of Cantor[23] and Poincaré,[24] to emphasize that the contradictions upheld in the philosophical field are resolved if it is believed that the continuum does not imply the admission of infinitesimals.

The infinitesimal calculus of Newton and Leibniz allows us to arrive at the theorem of Karl Weierstrass[25] for whom, precisely, the theory of sets is based on

18 Russell 2010, 351-359.
19 Russell 2010, 360-374.
20 Russell 2010, 330-335.
21 Aristotle 1936.
22 Russell 2010, 342.
23 Cantor 1895; 1897.
24 Poincaré 1900; Poincaré 1905.
25 Weierstrass 1894-1927.

the concepts of compactness and succession. The theorem of Weierstrass leads to Bernhard Bolzano,[26] whose work has contributed to broadening the horizon of mathematics on the side of logic. In addition to referring to his contemporaries, Russell does not neglect Zeno of Elea; he refers to the disciple of Parmenides, noting: “In this capricious world, nothing is more capricious than posthumous fame. One of the most notable victims of posterity’s lack of judgment is the Eleatic Zeno.”[27] Russell’s reference to the ancient pre-Socratic philosopher derives from the attention paid to the paradoxes of the movement, to substantiate the legitimacy of the theory of classes in these terms: “A class is a certain combination of terms, a class-concept is closely akin to a predicate, and the terms whose combination forms the class are determined by the class-concept. Predicates are, in a certain sense, the simplest type of concepts, since they occur in the simplest type of proposition.”[28]

Wishing to expound the theory of classes, Russell feels the need to look to an ancient philosopher, most often remembered as the “Eleatic Palamedes” (in Plato’s *Phaedrus*), that is to say as a faithful disciple of the master, whose only goal with his “speeches” was to reaffirm the legitimacy of the thesis on the uniqueness of being. In effect, Zeno’s paradoxes have a profoundly problematic nature of a philosophical kind, only resolved in the modern age in mathematics and logic with the theory of infinitesimal calculus. In the wake of the reworking by George Noël,[29] Russell pays attention to the four paradoxes on movement, among which the most problematic is the first, which seems to deny movement despite its phenomenal appearance. The concept of motion implies a reference to the pure forms of sensitive experience expounded by Kant, both in *De mundi sensibilis atque intelligibilis forma et principiis dissertatio* and in *Kritik der reinen Vernunft* (in particular in “Die transzendentale Ästhetik”). If we examine the physiology of the Pre-Socratics, in addition to Zeno’s paradoxes, we must refer to the concept of “infinity” developed by Anaximander. In Zeno’s case, the concepts of infinity and infinitesimal have a double value of a mathematical and logical nature, which found a solution with the theory of infinitesimal calculus developed by Newton and Leibniz at the end of the 17th century and with Russell’s class theory at the beginning of the 20th century.

Zeno’s paradoxes on motion were expounded in Aristotle’s *Physics.* For Aristotle the term “motion” possesses the broadest meaning of “change”; indeed, the “body in motion” indicates spatial motion, but it can be understood as that which changes over time. Therefore we should carefully reread the second part of Plato’s *Parmenides*, in relation to the third hypothesis, where the keyword “instant” appears. The first part of *Parmenides* suggests reflecting on the content of the paradoxes on movement developed by Zeno. Specifically, the most striking is the first, which Russell summarizes in these terms: “There is no motion, for what

26 Bolzano 1851.
27 Russell 2010, 352.
28 Russell 2010, 56.
29 Noël 1893.

moves must reach the middle of its course before it reaches the end."[30] Traditionally it is remembered as a regression *ad infinitum*, since the arrow, before getting from A to B, must reach an intermediate point C, and so forth. Arithmetically, according to Russell, Zeno's paradoxes on motion can be considered as a class of values from 0 to 1, for which the values of ½, ¼ and so on are assumed. If the whole is considered to be made up of an infinite number of terms, nothing prevents us from believing that in an infinite time it is possible to enumerate, one by one, the terms that compose it. Indeed, with the enumeration (or extensional) method we can represent a finite class; for an infinite class common sense too suggests resorting to the intensional method based on the concept-class. According to Russell, Zeno's paradox allows us to overcome the contradictions that have been noticed over time in relation to the concepts of *continuum* and *infinitum*, if we resort to class theory and the intensional method.

The intensional method allows us to overcome the apparent absurdity inherent in Zeno's paradoxes, especially when re-reading the first paradox on motion. The set of points that make up the line cannot be considered as a whole subsequent to the parts that make it up. Besides, Zeno did not intend to deny movement, but to problematize the explanation of decomposition to infinity, which in logical and mathematical terms was to be clarified with the theory of infinitesimal calculus developed by Newton and Leibniz and with the theory of classes conceived by Russell. Continuing to examine Russell's works, we find that the theory of classes undergoes a certain evolution with the theory of types, expounded, even before *Principia Mathematica* (written in the years 1910, 1912, 1913 with the collaboration of Whitehead), in some essays that appeared in the years immediately following the publication of *The Principles of Mathematics* (1903). Therefore it is worth mentioning a communication presented in 1905 to the London Mathematical Society "On the Substitutional Theory of Classes and Relations."[31] We also have to remember the following essays: "Les paradoxes de la logique"[32] which appeared in 1906 in *Revue de Métaphysique et de Morale*, "Mathematical Logic as based on the Theory of Types"[33] which appeared in 1908 in *American Journal of Mathematics* and "La Théorie des types logiques"[34] which appeared in 1910 in *Revue de Métaphysique et de Morale*. Russell's goal is to develop a logic free from antinomies, as anticipated in "Appendix B" of *The Principles of Mathematics*, to overcome the contradiction set out in the tenth chapter "with regard to predicates not predicable of themselves."[35] On the theory of types in "Appendix B" of *The Principles of Mathematics*, Russell emphasizes that "A *term* or *individual* is any object which is not a range. This is the lowest type of object."[36] Russell's problem of antinomies

30 Russell 2010, 353.
31 Russell 1906a.
32 Russell 1906b.
33 Russell 1908.
34 Russell 1910.
35 Russell 2010, 101.
36 Russell 2010, 534.

is presented in mature form in his 1908 essay ("Mathematical Logic as based on the Theory of Types"), where, precisely, he summarizes the most significant contradictions in the history of thought. It is not by chance that this 1908 essay on type theory was included in the 1956 collection that bears the title *Logic and Knowledge*,[37] set at the heart of the theory of knowledge. This collection of essays refers to the 1913 manuscript, *Theory of Knowledge*,[38] which was unpublished until 1992, after the critical encounter with the young Wittgenstein.

After the contradiction of Epimenides ("I'm lying"), Russell recalls the paradox of Jules Richard ("divisible only by unity and by itself") and Burali-Forte ("the set of all ordinal numbers"). The theory of types, in Russell's intention, is conceived to resolve the contradictions inherent in the system of traditional philosophy, to be overcome with a correct use of logical syntax. He does not ignore the paradox he himself conceived and expounded on the basis of the so-called "vicious circle"; hence criticizing Frege's mathematical framework, he objects that what comprises a set cannot be part of that set. That is to say "Whatever involves all of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total."[39] This contradiction is aimed at simplifying the laws of thought, avoiding "all propositions" or "all properties" statements. Russell objects that the "proposition all men are mortal" is an example of contradiction, since it would be false if no man existed. It would be more correct or less contradictory to assert that "it is always true that if x is a man, x is mortal."[40] In this case, the function of the time adverb "always" remains open, since x does not always correspond to a man. So we could say: "it is always true that x is mortal."[41] But in this case the proposition would be false. Consequently, it is necessary to focus on the value of the variable x, if it is understood as a man, or on the fact that each propositional function has a range of significance, or again on the fact that the function is true or false within a given field.

Russell amplifies the relevance of contradictions related to type theory; he believes that the problems posed involve the culture of the time, which in fact is evolving in other directions and with other interests. Therefore it could be said that the issues raised with *The Principles of Mathematics*, within his production, are central in relation above all to the concepts of *continuum* and *infinitum*. Confronting an ancient pre-Socratic philosopher, Russell was able to elaborate and resolve the contradictions apparently inherent in the paradoxes on movement, which from Zeno onwards had haunted entire generations of philosophers and men of science. The debate that took place from the development of infinitesimal calculus on leads to the contributions made by philosophy and science during the 19th century down to the neo-positivism developed at Trinity College

37 Russell 1956, 59-102.
38 Russell 1992.
39 Russell 1908, 225.
40 Russell 1908, 233.
41 Russell 1908, 233.

Cambridge. In any case, recognizing the merits given by Russell to twentieth-century philosophy and epistemology cannot lead us to neglect the richness of contents of the entire 20th century.

4. Final considerations

1903 was the year in which John Dewey published *Studies in Logical Theory*, Charles Renouvier *Le personnalisme*, and Henri Bergson *Introduction à la métaphysique*. In 1903 Wilhelm Nietzsche's *Die Philosophie im tragischen Zeitalter der Griechen* and Karl Marx's *Vorwort* a *Zur Kritik der politischen Ökonomie* were published posthumously. These are testimonies capable of enriching the social fabric of philosophy at the beginning of the 20th century, also reflected in the development of Edmund Husserl's transcendental phenomenology and Martin Heidegger's fundamental ontology. Both Husserl (who wrote *Logische Untersuchungen* in 1900-1901) and Heidegger (who wrote *Sein und Zeit* in 1927) confirm the richness of early twentieth-century philosophy, which was to find unprecedented outlets during the second half of the century. Continuing to dwell on the early 20th century, it should be emphasized that the development of neo-positivism found an almost natural outlet in the logical empiricism of the Vienna Circle and the Berlin Circle and in the attempt to found a true scientific philosophy, as we can read in *Die wissenschaftliche Konzeption der Welt*, published in 1929. As an original and personal link between Cambridge and the European continent, we must underline Ludwig Wittgenstein's "Logisch-philosphische Abhandlung", which appeared in 1921 in Wilhelm Ostwald's *Annalen der Naturphilosophie*; the following year it was published in English with the title *Tractatus logico-philosophicus* with an "Introduction" by Russell.

On the specific plane of science, Albert Einstein's theory of relativity and Max Planck's and Werner Heisenberg's quantum theory are worth mentioning. On the specific plane of epistemology, some cultural orientations cannot be overlooked: for example, the empirio-criticism of Richard Avenarius and Ernst Mach; the conventionalism of Jules-Henri Poincaré and Pierre-Maurice-Marie Duhem. Also worthy of note are Gaston Barchelard's rational materialism, Thomas Samuel Kuhn's scientific relativism, Imre Lakatos' falsificationism, and Paul Karl Feyerabend's anarchist method. Critical rationalism and the falsifiability principle of Karl Raimund Popper constitute a separate chapter, but we must not forget the developments of non-Euclidean geometries with the contributions of Bernhard Riemann and Nikolai Ivanovich Lobachevsky, in the wake of the essay that Poincaré in 1891 specifically dedicated to *Les géométries non euclidiennes*. In the context of the new sciences, it is necessary to emphasize Eugenio Beltrami's *Saggio di interpretazione della geometria non euclidea* (1867), confirming the intense debate that was already taking place during the second half of the 19th century on the plane of philosophy and science or, if we prefer, science and philosophy. Federigo Enriques made important contributions to science into this cultural panorama, in particular with his work on *Problemi della scienza*, published in 1906.

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