JOURNAL OF INTERDISCIPLINARY HISTORY OF IDEAS

2022

Volume 11 Issue 22
Item 4

– Section 2: Articles –

The Hybridization of Practical and Theoretical Geometry in the Sixteenth-Century Euclidean Tradition

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JIHI 2022

Volume 11 Issue 22

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This article aims to show how, in the sixteenth century, Euclidean geometry, which was regarded as the epitome of theoretical geometry in the middle ages and in the Renaissance, was to take up, within certain printed commentaries and translations of Euclid's Elements, *features that were typical of practical geometry and how this contributed to the development of an approach to geometry, and also to a representation of geometry, that may be regarded as a hybrid of theoretical and practical geometry within the Euclidean context.*

1. Euclidean geometry and practical geometry in medieval and early modern Europe

1.1. Euclid's Elements and its status in pre- and early modern mathematical culture

The *Elements* of Euclid had become by the seventeenth century a true best-seller. In fifteen books,[1] this treatise dealt with the properties and constructions of plane magnitudes (Books I-IV), the theory of ratios and proportions

* The research that made this article possible has been carried out at the MPIWG (Berlin) in the framework of the project *Making Euclid practical: The impact of practical geometry on the Euclidean tradition in the sixteenth century* funded by the Gerda Henkel Stiftung (2021–2023). A part of the research concerning the practical geometry tradition was carried out in 2017–2018 in the framework of a postdoctoral fellowship at the History of Science department of the Technische Universität Berlin.
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1 That is, including the two apocryphal books on regular polyhedra by Hypsicles of Alexandria and by Isidore of Miletus which were generally published with Euclid's thirteen books in the sixteenth century.

Journal of Interdisciplinary History of Ideas 11(2022), 22, p. 4:1–4:104. Peer-reviewed.

applied to magnitudes and the constructions and relations of similar figures (Books V-IV), number theory and the theory of ratios and proportions applied to numbers (Books VII-IX), incommensurable magnitudes (Book X) and the geometry of solid figures (Books XI-XIII), to which were added two apocryphal books on the regular polyhedra (Books XIV-XV). All these mathematical objects were dealt with abstractly, that is, without any reference to matter, instruments, or to any exterior applications of geometry or arithmetic. Their properties and modes of constructions were demonstratively derived, in the context of theorems and problems, from self-evident or previously admitted principles (definitions, postulates and common notions) or from previously demonstrated propositions.

Because of the abstractness of its objects, as well as its deductive and axiomatic structure, Euclid's *Elements*, and most of all its geometrical books, was held to possess the argumentative form, rigor, necessity and universality which was conform to the notion of science based on Aristotle's *Posterior* analytics. It came, as such, to represent in the premodern era a model to follow in any scientific endeavor.[1] Also, the fact that Euclid's *Elements* started with the most fundamental geometrical objects to end with the properties of regular polyhedra enabled the geometry of Euclid to be considered in learned and lay scientific culture (notably in the mathematical curricula,[2] in the visual representations[3] and in artisanal contexts[4]) as the canon of geometrical knowledge, representating the science of geometry in general. In the sixteenth century, Euclid's *Elements* was held to encompass all the knowledge necessary to allow a perfect understanding and practice of mathematics,[5] wherefore it was edited, translated

[1] Axworthy 2016, chap. 2, in part. 90-94 and Higashi 2018, 81-121.

[2] Weijers (1996, 14 and 26) and Høyrup 2014.

[3] See for example the visual representation of the division of philosophy in Gregor Reisch's *Margarita philosophica* (Reisch 1504, 2v), in which geometry is associated with Euclid, and the frontispiece of Samuel Marolois's *Opera mathematica* (Marolois 1614), in which geometry is represented by Euclid while Archimedes represents military architecture and engineering, analysed in Knobloch 2005 and Remmert 2008, 541.

[4] L. Shelby (in Shelby 1972) showed how "for mediaeval masons Euclid had virtually become an eponymous hero of the craft (geometry)" (p. 396), even if masons rarely had any contact with Euclid's geometrical work and identified geometry with masonry.

[5] This is shown, for instance, in the frontispiece of Niccolò Tartaglia's *Nova scientia* (1537), in which Euclid is represented as guarding a gate leading to a numerous group of allegorical figures

and commented on a considerable number of times and by a large number of different authors,[1] among whom were humanists, school masters or university lecturers, as well as learned artisans such as architects.

1.2. The origin and scope of medieval practical geometry

In the sixteenth century, the multitude of printed editions of the *Elements* was rivalled by the growing number of printed treatises of practical geometry. Practical geometry took its origin in a multitude of traditions, from the metrological corpus of Hero of Alexandria, Roman *agrimensura* and Arabic treatises of applied geometry to the mathematical textbooks of clerical schools and Euclidean geometry.[2] However, it was explicitly presented as a type of geometrical teaching in its own right from the twelfth century with Hugh of Saint-Victor's *practica geometriae*.[3] In Hugh of Saint-Victor's treatise, practical (or active) geometry was explicitly distinguished from theoretical (or speculative) geometry and was presented as an art of measuring by instrumental means:

representing the mathematical sciences. See Valleriani 2013, 61.

1 The following list only takes into consideration the first edition of each work. Rare were those that were not reedited or reprinted at least once. Campanus (Venice: Ratdolt 1482), Valla (Venice: Bevilaqua 1498), Zamberti (Venice: Tacuino 1505), Pacioli (Venice: Paganini 1509), Lefèvre d'Étaples (Paris: Estienne 1516), Voegelin (Vienna: Singrenius 1528), Politi (Siena: Nardi 1529), Grynaeus (Basel: Hervagius 1533), Fine (Paris: De Colines 1536), Tartaglia (Venice: Roffinelli 1543), Caiani (Rome: Blado 1545), Ramus (Paris: Grandin 1545), Camerarius (Leipzig: Rheticus 1549), Scheubel (Basel: Hervagius 1550), Montdoré (Paris: Vascosan 1551), Benedetti (Venice: Caesano 1553), Nabod (Cologne: Birkman 1556), Magnien-Gracilis (Paris: Cavellat 1557), Peletier (Lyon: Tournes & Gazeau 1557), Vinet (Bordeaux: Millanges 1559), Xylander (Basel: Oporinus 1562), Forcadel (Paris: De Marnef & Cavellat 1564), Dasypodius (Strasburg: Mylius 1564), Dasypodius and Herlinus (Strasburg: Mylius 1566), Sthen (Wittenberg: Seitz 1564), Foix-Candale (Paris: Le Royer 1566), Billingsley (London: Daye 1570), Commandino (Pesaro: Francichini 1572), Clavius (Rome: Accolto 1574), Zamorano (Sevilla, La Barrera: 1576), Errard (Paris: Auvray 1598), Dybvad (Arnhem: Jansson 1603), Dou (Leyden: Bouwensz 1606). On the sixteenth-century Euclidean tradition, see Høyrup 2019.

2 Shelby 1972; L'Huillier 1994; Raynaud 2015, 9-10. The possible influence of twelfth-century Hebrew sources on the Latin medieval practical geometry tradition has been surmised in Friedman and Garber 2022. On this issue, see also Homann 1991, 18 and Corry 2013.

3 Baron 1955. Before the twelfth century, elements of practical geometry were taught together with notions of Euclid geometry, notably within the curricula of the cathedral schools. On this tradition, see Høyrup 2014, as well as Evans 1976 and Zaitsev 1999. In Hugh of Saint-Victor's *Didascalicon*, the entire subject-matter of geometry corresponds to the object of practical geometry such as described in the *Practica geometriae*, since he then writes that "Geometry has three parts: planimetry,

Geometry is either theoretical (speculative) or practical (active). Theoretical geometry uses sheer intellectual reflection to study spaces and intervals of rational dimensions. But practical geometry uses instruments, and gets its results by working proportionally from one figure to another.[1]

After Hugh of Saint-Victor, several other medieval authors wrote on the topic, such as Leonardo Fibonacci (or Leonardo of Pisa), John of Muris and Dominicus de Clavasio. Fibonacci's *Practica geometriae* dealt with measurement and divisions of determinate magnitudes through more diverse techniques, mostly computational.[2] John of Muris' *De arte mensurandi*, beyond techniques to measure and divide magnitudes, included instructions to construct figures, as well as elements of trigonometry, and arithmetical and algebraic rules[3] and Dominicus de Clavasio's fourteenth-century *Practica geometriae* taught both instrumental and non-instrumental techniques to measure lengths, areas and volumes on the basis of the theory of proportions.[4]

Many Renaissance practical geometry treatises can be placed in the continuity of this medieval tradition, such as Oronce Fine's *Geometria practica* (first published as the second book of the *Geometria* in his 1532 *Protomathesis*),[5] Leonard and Thomas Digges' *Pantometria* (1571)[6] or Jean Errard's *Geometrie et practique*

altimetry, and cosmimetry" (Homann 1991, 34). S. J. Victor (in Victor 1979, 4-7) surmised that the *Didascalicon* was written before the *Practica geometriae* and that, between the compositions of these two works, Hugh would have had access to the text of Euclid's *Elements* through the first Latin translation made Adelard of Bath (probably written in the second quarter of the twefth century), which would have motivated him to clearly distinguish practical and theoretical geometry in the latter work.

1 Hugh of Saint-Victor (Baron 1966, 16; transl. Frederick A. Homann 1991, 33-34): "considerandum est quod omnis geometria disciplina aut theorica est, id est speculativa, aut practica, is est activa. *Theorica siquidem est que spacia et intervalla dimensionum rationabilium sola rationis speculatione vestigat, practica vero est que quibusdam instrumentis agitur et ex aliis alia proportionaliter coniciendo diiudicat*" (my emphasis). On this distinction between practical geometry and theoretical geometry, see Baron 1955; Shelby 1972; Victor 1979, 3-7; Moyon 2008, 140-141.

2 Boncompagni 1862, 1; tr. Hughes 2008, 4; Victor 1979, 26-27 and 47-49; Moyon 2017, 111-113.

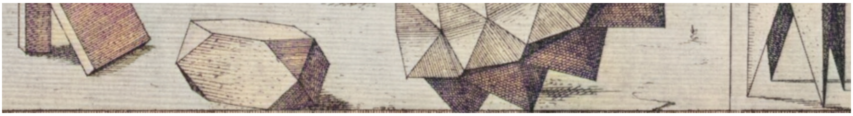
3 Victor 1979, 49-51; Busard 1998; Moyon 2017, 114-116.

4 Busard 1965 and Victor 1979, 29-30 and 51-52.

5 Fine 1532, 64r-99r; Fine 1544. On the status of practical geometry in Fine's mathematical work, see Métin 2004, Brioist 2009a and Axworthy 2016, 266-280.

6 Digges 1571.

generalle d'icelle (1594),[1] which all took up Hugh of Saint-Victor's subdivision of practical geometry in the measure of lengths, surfaces and volumes (i.e. in *altimetria*, *planimetria* and *cosmimetria*,[2] the latter being later expressed through the more general term *stereometria*[3]).



1.3. The distinction between practical and theoretical geometry in the middle ages

If practical geometry and Euclidean geometry can be ultimately considered as belonging to the same branch of mathematical knowledge (both dealing with continuous magnitudes and bearing the name geometry), if a part of the content of practical geometry treatises was indirectly derived from Euclidean material[4] and also if Euclidean geometry could itself be regarded as possessing a practical

[1] Errard 1594. On Errard's practical geometry, see Métin 2016, I, 233-268.
[2] Saint-Victor 1966, 17. Fine 1532, 64r: *Liber secundus Geometriæ, de practicis longitudinum, planorum & solidorum, hoc est, linearum, superficierum, & corporum mensionibus, alijsve mechanicis, ex demonstratis Euclidis elementis corolarius: ubi et de quadrato geometrico, et virgis seu baculis mensorijs.* (The three parts of practical geometry are then divided in *Longimetra*, *Planimetra*, *Profundimetra*); Digges 1571, title page: *A Geometrical Practise, Named Pantometria Divided into Three Bookes, Longimetra, Planimetra, and Stereometria, Containing Rules Manifolde for Mensuration of all Lines, Superficies and Solides*; Errard 1594, 1, 13 and 45: "Le premier livre de geometrie de J. Errard de Bar-le-Duc. De la mesure des lignes droites. Et premier de la composition de instrument. (...) Le second livre de la mesure des Superficies planes. (...) Le troisiesme livre de la mesure des Solides".
[3] In Hugh of Saint-Victor's *Practica geometriae*, cosmimetria, which literally means the measure of the world, aims to deal with the measurement of celestial distances to be specifically applied to spherical astronomy. The notion of *stereometria* thus goes beyond this aim by considering the measures of all types of bodies.
[4] Shelby 1972, Evans 1976, L'Huillier 1994 and Raynaud 2015, 15-16.

aspect in the form of *problemata*,[1] a clear distinction between practical and theoretical geometry was nevertheless made between them in the introductions of medieval practical geometry treatises,[2] starting with Hugh of Saint-Victor's *Practica geometriae*, but also when introducing to Euclid's *Elements*, as in the preface of Johannes de Tinemue (or John of Tynemouth) to his commentary on Adelard of Bath's translation of the *Elements* (known as Adelard III) and which dates from the late twelfth or early thirteenth century.[3] According to this preface, while the aim of theoretical geometry would be to carry out demonstrations concerning magnitudes, practical geometry would aim to measure magnitudes by the means of determined measuring units, such as the perch, the palm, the inch and the foot (*pertica*, *palma*, *digitus*, *pes*).[4] Euclid's geometry was then held to belong to theoretical geometry insofar as the *modus agendi* of the author of the *Elements* is to argue demonstratively from true and first principles.[5] Indeed,

[1] As defined by Proclus, in his commentary on Euclid, the aim of problems is to teach how to construct or find a certain geometrical object, while theorems aim to demonstrate a universal property or relation of geometrical objects. Proclus (Friedlein 1873, 77; transl. Morrow 1992, 63): "The propositions that follow from the first principles he divides into problems and theorems, the former including the construction of figures, the division of them into sections, subtractions from and additions to them, and in general the characters that result from such procedures, and the latter concerned with demonstrating inherent properties belonging to each figure. Just as the productive sciences have some theory in them, so the theoretical ones take on problems in a way analogous to production". On the practical scope of problems, see *infra*.

[2] On the medieval practical geometry tradition and the distinction between theoretical and practical geometry, see Shelby 1972, Victor 1979, 2-5, L'Huillier 1994, Moyon 2008, I , 140-151, Raynaud 2015, 9-11.

[3] Busard 2001. On this commentary, see Murdoch 1968 and Busard 2001, 12-16.

[4] Busard 2001, 31-32: "Sapientia in theoricam et practicam. (...) Ecce mathematica continet geometriam quasi genus. Genus denique secundum materialem suppositionem est magnitudo circa cuius species postea versatur. Artifex vero est tam demonstrator quam exercitator. Officium demonstratoris est ad intelligentiam discipline theoremata explicare (...). Officium exercitatoris est mensurare. Est autem mensuratio certe quantitatis assignatio. Instrumentum vero demonstratoris est radius et mensa cum pulvere. Exercitatoris vero instrumenta sunt mensure geometrice scilicet pertica cum palma, digitus, pes, passus et ulna".

[5] Busard 2001, 33: "Intentio auctoris est rationalium figurarum mensurationem expedire. (...) Modus agendi is est: Agit namque demonstrative. Est autem demonstratio argumentatio, arguens ex primis et veris vel illorum conclusionibus. Sic enim ars proposita contexta est quod sequentia accidunt ex premissis necessario aut principiis deinceps. Est enim demonstrativa scientia que docet et demonstrare et demonstrat ut posteriores analeti. Et que demonstrat et non docet demonstrare ut geometria".

in Tinemue's language, the actor of theoretical geometry corresponds to the *demonstrator*, which is then distinguished from the *exercitator*, or the actor of practical geometry.[1]

We also find an explicit distinction between theoretical and practical geometry in the philosophical work of Dominicus Gundisalvi,[2] where this distinction is based on the distinction between practical and theoretical knowledge internal to the division of philosophy.[3] Hence, in his *De divisione philosophiae*, Gundisalvi wrote that "the end [or goal] of theoretical geometry is to teach something; while the end of practical geometry is to do something", founding the distinction between theoretical and practical knowledge on the opposition between knowing and doing. He defined "the artificer of theoretical geometry" as "the geometer who has become acquainted thoroughly with all parts of geometry and teaches it" and "whose instrument is demonstration", while "the artificer of practical geometry is he who employs it in working".[4] Now, the examples Gundisalvi used to present theoretical geometry in more detail are drawn from Euclid's *Elements*,[5] which indicates that Euclid's geometry was held in this context as chiefly representative of theoretical geometry. Practical geometry, on the other hand, is associated with the activities of the measurers, who "measure the height, the depth, or the level surface of the Earth" with various units

[1] Busard 2001, 33: "Artifex vero est tam *demonstrator* quam *exercitator*. (...) Officium *demonstratoris* est ad intelligentiam discipline theoremata explicare (...) Officium *exercitatoris* est mensurare".

[2] Gundisalvi's classification of mathematics and characterisation of geometry takes up much of its content from al-Farabi's *Enumeration of the sciences*, of which Gundisalvi's *De scientiis* constitutes an adaptation. On Gundissalvi's classification of sciences, see Weijers (1996, 190-191) and on the sources of his *De divisione philosophiae*, see Hugonnard-Roche (1984).

[3] Gundisalvi (Baur 1903, 12 and 44): "Partes igitur, in quas primum philosophia dividitur, hee sunt: scilicet theorica et practica (...) Cum autem omnis ars dividatur in theoricam et practicam, quoniam vel haberetur in sola cognicione mentis – et est theorica –, vel in exercicio operis – et est practica –: profecto ars extrinsecus pertinere videtur ad theoricam, ars intrinsecus ad practicam. Ars enim extrinsecus non tradit actum, set scienciam tantum; ars vero intrinsecus et actum dat et scienciam". Thus, geometry, as each part of mathematics (at least arithmetic, music, geometry, astronomy), possesses a theoretical and a practical part: (Baur 1903, 104): "geometria alia est practica, alia theorica".

[4] Gundisalvi (Baur 1903, 107-109); transl. in Grant 1974, 72.

[5] Gundisalvi (Baur 1903, 106-107): "Species theorice sunt tres, scilicet operacio, sciencia, invencio (...) ad agendum propununtur ut gracia exempli primum et secundum theorema Euclidis (...). Ad sciendum vero proponuntur ut quintum theorema Euclidis (...). Ad inveniendum autem ponuntur ut primum theorema tercii libri".

of measurement such as the foot or the cubit (as in Tinemue's preface to Euclid's *Elements*), and of the artisans, who

> exert themselves in manufacturing or in working in the mechanical arts, as a carpenter works on wook, an iron worker on iron, a stone mason on cement and stones, and similarly every artificer of the mechanical arts works according to practical [or applied] geometry.[1]

Roger Bacon offered an even more extensive definition of practical geometry in his *Communia mathematica*, defining it as dealing with mathematical instruments, measurements and all other useful utilitarian applications, from surveying to mechanics, optics, architecture and engineering.[2] While distinguishing practical geometry from theoretical geometry (which included Euclid's *Elements* as well as other works[3]), he asserted the priority of the latter on the former in the order of learning, dignity and utility, given that it exhibits the causes of the instruments and operations at play within the material world and that are useful to human life.[4]

1.4. The distinction between practical and theoretical geometry in the sixteenth century

In the sixteenth century, practical geometry was often clearly distinguished from theoretical geometry in the introductions of practical geometry treatises,

[1] Gundisalvi (Baur 1903, 109); transl. in Grant 1974, 72.

[2] Bacon 1940, 42-47: "Practica vero Geometrie descendit ad instrumenta scienciarum et mensuraciones et cetera opera utilia in rebus utilibus (aliter civilibus) et habet octo partes magnas". See Victor 1979, 37-39 and Raynaud 2015, 11.

[3] Bacon 1940, 41-42: "apud Latinos non est tradita speculativa ab uno auctore nec in uno volumine sed partim in libro *Elementorum* Euclidis vulgato, et in libro eius *de Quantitatibus Datis*, et in libris eciam Theodosii *de Speris*, et in libro Jordani *de Triangulis* (...)".

[4] Bacon 1940, 41-42: "Geometria vero speculativa est prior quam sua practica, quia operacio addit supra nudam speculacionem, et universaliter finis speculative est practica, et difficilior et nobilior et longe utilior, sicut finis se habet ad ea que sunt ad finem. Geometria igitur speculativa docet omnes partes quantitatis continue intrinsece, et omnes partes et omnes passiones propria earum et omnes causas abseque eo quod descendat ad instrumenta et operaciones et cetera opera utilia in hoc mundo".

as in Gregor Reisch's *Margarita philosophica* (1503),[1] Fine's *Geometria practica*[2] and Jacques Chauvet's *Pratique universelle de geometrie* (1585).[3] In these texts, practical geometry was, as in the medieval texts presented above, mostly defined as an art of measuring concrete lengths, areas and volumes[4] and theoretical geometry was explicitly identified with the geometry of Euclid.[5]

[1] Reisch 1504, r2r: "In praxim Geometriae tractatus secundus. *Dis.* Ex dictis nondum intelligo quo nam pacto aut ingenio terram spaciumve terrae metiri possim quod tamen geometriae nomen visum est insinuare. *Mag.* Verum dicis: nam tibi in prioribus geometriae generaliora principia, paucasque eiusdem theorias proposui. Iam vero quia te vigilantissimum, eiusque percupidum sentio: de metiendi modo pauca subiungere aenitar. In primis autem quid mensura: & quae eius partes praemittam".

[2] Fine 1532, 64r: "Duo sunt, optime lector, quæ in omni disciplina, studiosis omnibus solent esse non iniucunda. unum est, facilis in disciplinam introductio: qua & via doctrinæ, & sensus eiusdem universus aperitur. Reliquum esse videtur, collectus ex ipsa disciplina fructus, susceptorum laborum compensator gratissimus. Præmissis itaque generalibus ipsius Geometriæ rudimentis, ad elementorum Euclidis, & succedentium nostrorum operum intelligentiam isagogicis: consequens nobis visum fuit, universam Geometriæ subnectere praxim, hoc est, linearum, superficierum, & corporum, ex demonstratis Euclidis elementis, ostendere mensuram".

[3] Chauvet 1585, 1r: "Geometrie est une science qui considere la quantité continue: laquelle est divisee en Theorique & Pratique. Et pour autant que la Theorique est suffisamment traictée & demonstrée aux Elemens d'Euclide, nous en ferons un tacet, & renvoirons le nouveau aprentif à iceux principes, sans lesquels il ne peult parvenir à la vraye cognoissance de la Geometrie Pratique: laquelle est subdivisee en trois especes, sçauoir Longuemetre, Planemetre, & Stereometre ou Solidimetre".

[4] Reisch 1504, r2r-v: "In praxim Geometriae tractatus secundus. *Dis.* Ex dictis nondum intelligo quo nam pacto aut ingenio terram spaciumve terrae metiri possim quod tamen geometriae nomen visum est insinuare. (…) de metiendi modo pauca subiungere aenitar. (…) de qua ea geometriae pars est: quam altimetram vocant; quae in sursum & deorsum & illa quam planimetram dicunt: quae autem secundum in ante & retro, dextrorsum & sinistrorsum mensurat"; Fine 1532, 64r: "universam Geometriæ (…) praxim, hoc est, linearum, superficierum, & corporum (…) mensuram".; Chauvet 1585, 1r: "la Geometrie Pratique: laquelle est subdivisee en trois especes, sçauoir Longuemetre, Planemetre, & Stereometre ou Solidimetre". See also Chauvet's dedicatory epistle to Anne d'Anglure, in Chauvet 1585, a2v: "Il n'est besoin de vous dire quelle utilité la Noblesse peut recevoir de ceste Pratique, puisque ce vous est une chose tant cognue & laquelle vous avez tousjours sagement estimee estre necessaire à tout Gentil-homme qui veut faire profession de commander és Armees, soit pour faire un Pont à faire passer la Gendarmerie, soit pour combler un fossé, ou sçavoir sur le champ de quelle hauteur & largeur doit estre une Bresche pour recevoir vingt hommes de front".

[5] Reisch 1504, p7r: "Geometriae elementa & principia abunde tradidit & conscripsit Euclide pater"; Fine 1532, 64r: " in disciplinam introductio: qua & via doctrinæ, & sensus eiusdem universus aperitur (…) generalibus ipsius Geometriæ rudimentis, ad elementorum Euclidis"; Chauvet 1585, 1r: "la Theorique est suffisamment traictée & demonstrée aux Elemens d'Euclide". On the dichotomy

In the practical geometry treatises in which only practical geometry was presented, such as the *Pratique de Geometrie* of Jean de Merliers (1575), the *Geometria prattica* of Giovanni Pomodoro (1599) and the *Geometria practica* of Christoph Clavius (1604), practical geometry is then again identified with an art of measuring concrete or dimensioned objects, lengths, areas and volumes, as in Clavius,[1] or mainly surfaces and the lines within them, as in Pomodoro[2] or Merliers. In the latter's text, practical geometry is specifically identified with surveying.[3]

Yet, as was made clear by Niccolò Tartaglia, in his 1556 *General trattato de numeri et misure*, practical geometry was not restricted to an art of measuring, since it also included a part that did not include any measurement by numerical

between theoretical and practical geometry in the sixteenth century, see Guyot and Métin 2004, Knobloch 2005 and Axworthy 2016, 255-257, for the specific case of Oronce Fine.

[1] Clavius 1604, p. 1-3: "Quandoquidem Mathematicarum disciplinarum stadium scribendo ingressi, nonnullam eius partem, favente Deo, percurrimus: *Geometriae practicae* tractatio omittenda non fuit (...) Etenim dum certa ratio traditur, qua *camporum longitudines, altitudines montium, vallium depressiones, locorum omnium inaequalitates inter se, & intervalla deprehendere metiendo* debeamus: cuilibet liquet, ut arbitror, quantum *commodi, utilitatisque substructioni aedificiorum, cultui agrorum, armorum tractationi, contemplationi siderum, aliisque artibus*, & disciplinis ex horum cognitione manare possit. Haec enim una Mathematicarum rerum scientiae pars, sicut ab artificibus ob sui necessitatem avidè semper est arrepta: ita ob insignes utilitates, quas in re tota militari suppeditat, in maximorum Principum, Regumque aulis omni tempestate versata est. (...) decrevi, si quà possem, perficere: ut, quicquid utiliter in *Geometria practica* ab aliis traditum, à me etiam inventum est, unius operis gyro clauderetur. Quod opus, cum species tres quantitatis continuae sint, in tria membra, partesque praecipuas secuimus: *In prima rectas lineas, in altera superficies, corpora metientes* in postrema: cui annectuntur alia, qua: non tam ad quantitatis dimensionem, quam ad alias Geometria praxes, ac demonstrationes pertinent, à nostro instituto non aliena." (My emphasis.

[2] Pomodoro 1599, Table VI (commentary by Giovanni Scala): "In questa sesta Tavola l'Autore ci comincia hora à insegnare *la practica della Geometria,* perche propone in essa *figure, le quali sono misurate con numeri*" ; Table VI: "Misurare in piu modi praticalmente li quadrati per numeri sani, & sani è roti"; Table VII: "Lato 12 s'adimanda il Diametro Praticalmente" or Table XIII: "Modi diversi geometrici, et prattici per trovare la superficie delle figure quadrilatere, dette doppi capitagliati".

[3] Merliers 1575, A2r: "Cognoissant Monseigneur, le grand prouffit & utilité, que *la Geometrie* apporte à l'homme, J'ay mis tout mon estude à faire une briefve & facile description de *la Practique d'icelle, qui contient le moyen d'Arpenter, c'est à dire de mesurer exactement toutes Superficies closes & bornées de lignes droictes, ou costez droictʒ*".

means and was purely geometrical,[1] which corresponds to what L. Shelby[2] designated as “constructive geometry”, which is mainly useful to master masons and artisans in the production of artefacts. Moreover, for Tartaglia, the first part of practical geometry, which deals with measuring practices, is itself subdivided in two categories, the minor or lower type, which deals with properly utilitarian problems involving the measurement of magnitudes and which is primarily addressed to surveyors, and the major or higher type of computational practical geometry, which differs from the minor or lower type in the fact that it deals with more speculative and less utilitarian matters.[3] Now, as announced in the same passage, Tartaglia intends to deal with each of these three types or subtypes of practical geometry in his treatise, part III dealing with the lower part of mensorial practical geometry, part IV dealing with the higher part, and part V dealing with constructive geometry.

In the Italian translation and commentary on Euclid’s *Elements* Tartaglia first published in 1543, which is one of the few commentaries on the *Elements* in which a clear distinction is made between practical and theoretical geometry, practical geometry is however mainly identified with the properly geometrical or constructive type, since its aim is to teach how “to draw, construct and manually make all things that are necessary”, being thus presented as relevant mainly to artisans.[4]

[1] Tartaglia 1556 III, 1r: “Delle specie della pratica geometrica. La general pratica geometrica dividemo in due parti, over specie, la prima delle quali è quella, che nelle gran quantita, & figure, si mescola con la pratica arithmetica, cioe la si denomina, de rappresenta con numeri di misure lineali, over superficiali, over corporee, & la seconda, laqual è pura geometrica non è mista con numeri, come al suo luogo s’intendera.”

[2] Shelby 1972.

[3] Tartaglia 1556 III, 1r: “Anchor la detta pratica mista con numeri dividemo in due specie. L’una chiamiamo pratica menore, & l’altra maggiore, la menore è la piu humile, over bassa, ma la piu utile, & necessaria a ogni qualita di persone, perche quella ne da il modo, & la regola da saper conoscer con numeri, & misure la quantita si corporea, come superficiale di tutte quelle cose, che manualmente misurar si possa (essendo personalmente sul fatto) & di questa sorte di pratica si ha da trattar (naturalmente parlando) in questa nostra terza parte, delle altre due specie, nelle due sequenti parti abondantemente ne parleremo.”

[4] Tartaglia 1543, 3v: “Anchora inanzi che piu oltra procediamo bisogna notar qualmente la scientia di Geometria, & di Arithmetica se divide in due specie, una del lequal (come fu detto in principio) é detta Theorica, cioe, speculativa,over contemplativa: l’altra è detta prattica, cioe, attiva, over operativa. La theorica, cioe, la speculativa (come afferma Ptolomeo nell’Almagesto) è per augmento

In this context, theoretical geometry is presented as aimed towards the production of knowledge, allowing "to continually discover new things and to augment science".[1] But in the *General trattato*, where practical geometry is presented as more diverse in nature than in the commentary on Euclid, theoretical geometry is said to "investigate the proximate causes of the other [i.e. the practical part of geometry], to consider and examine quantities, proportions and their measures by a speculation of the mind, and Euclid of Megara speaks about this [type of geometry] and deals with it in twelve books."[2] Theoretical geometry is thus clearly identified here with the type of geometry taught in Euclid's *Elements.*

Juan Perez de Moya, in the introduction of his 1573 *Geometria practica y speculativa* (which was based for a great part on his 1568 *Geometria pratica*),[3] did not at first describe the object of practical geometry with precision, but he clearly distinguished it from theoretical geometry, which, on the other hand, was explicitly identified with Euclidean geometry.[4] Practical geometry is then said to

della scientia, perche per mezzo della speculativa possiamo ritrovar continuamente cose nove, & ampliar la scientia. Ma la prattica, cioe, la operativa è per operar, cioe, per designare, construer, & fabricar manualmente tutte le cose occurrente." Interestingly, this part was suppressed in the 1565 edition.

[1] Tartaglia 1543, 3v: "Anchora inanzi che piu oltra procediamo bisogna notar qualmente la scientia di Geometria, & di Arithmetica se divide in due specie, una del lequal (come fu detto in principio) é detta Theorica, cioe, speculativa, over contemplativa: l'altra è detta prattica, cioe, attiva, over operativa. La theorica, cioe, la speculativa (come afferma Ptolomeo nell'Almagesto) è per augmento della scientia, perche per mezzo della speculativa possiamo ritrovar continuamente cose nove, & ampliar la scientia. Ma la prattica, cioe, la operativa è per operar, cioe, per designare, construer, & fabricar manualmente tutte le cose occurrente." Interestingly, this part was suppressed in the 1565 edition.

[2] Tartaglia 1556, III, 1r: "Delle specie della Geometria. Le specie principali della geometria sono due, delle quali l'una è detta theorica, & l'altra pratica. La theorica è quella che per investigare le propinque cause de gli effetti di quella, considera, & guarda le quantita, le proportioni, & le misure di quelle, con una speculatione di mente, & di questa abondantemente ne parla, & tratta Euclide Megarense in dodici libri." Euclid is here said to deal with theoretical geometry in twelve books since he counts the number of books in the *Elements* when including the two apocryphal books by Hypsicles and Isidore of Miletus, which make altogether fifteen books, minus the three books pertaining to arithmetic (Books VII–IX).

[3] On Perez de Moya's mathematical works, see Silva 2013.

[4] Perez de Moya 1573, 5: "Dividese [la geometria] como las demas artes en Theorica, y en Practica. *La Theorica, ò Speculativa es aquella*, que por hallar la causa de los effectos de la Practica, consid-

"deal with putting into effect or implementing the reasons on which the mind reflects in theoretical geometry", marking its operative character and its dependence on theoretical geometry. Yet, after distinguishing in such way practical and theoretical geometry, and after briefly mentioning the three different types of magnitudes considered in geometry (line, surface, solid), he distinguished the three parts of the practice of measurement into *altimetria*, *planimetria* and *stereometria*, which (as he announced then) will be treated respectively in Books 2, 3 and 4, [1] that is, after presenting, in Book 1, the more theoretical geometrical principles (i.e. mainly definitions, postulates and common notions drawn from Euclid's *Elements*) which are necessary to the apprehension of the following books.[2]

In a different manner, Jean Bullant, in his 1562 *Petit Traicté d'horologiographie et geometrie pratique*, distinguished the "speculative and theoretical part" of geometry taught by Euclid from a geometry familiar to artisans.[3] In the preface

era la quantidad, y proporcion con una especulacion del entendimiento, *de lo qual trato Euclides compendiosa y cumplidamente*, y nosotros en este tratado diremos lo que hiziere al proposito, para entendimiento de lo que en nuestras obras pretendemos dezir. *La Practica trata, de poner en effecto, ò en obra las razones que el entendimiento en la Theorica Speculo*, de la qual trataremos en los otros libros siguientes desta obra" (my emphasis). When Perez de Moya writes: "La Theorica, ò Speculativa es aquella, que por hallar la causa de los effectos de la Practica, considera la quantidad, y proporcion con una especulacion del entendimiento, de lo qual trato Euclides compendiosa y cumplidamente", he seems to be taking up the definition of theoretical geometry that Tartaglia presented above when he wrote (Tartaglia 1556 III, 1r): "La theorica è quella che per investigare le propinque cause de gli effetti di quella, considera, & guarda le quantita, le proportioni, & le misure di quelle, con una speculatione di mente, & di questa abondantemente ne parla, & tratta Euclide Megarense in dodici libri".

[1] Perez de Moya 1573, 5: "Los generos de las medidas son tres. *Altimetria, Planimetría, Stereometria. Altimetria* trata de la orden de medir las cosas segun sus anchuras, ò alturas, ò larguras solamente. En este genero entra el medir distancias, profundidades, y alturas, *comos veras en el lib. 2. Planimetria*, trata de medir lo superificial de los cuerpos de qualquiera suerte que sea. En este genero entra el medir campos, ó heredades para saber la quantidad de hanegas de pan que en ellas se pueden sembrar, *como trata el 3 lib. Stereometria*, trata de medir las cosas segun su largor, y anchor, y profundidad. En este genero se incluyen las medidas de lo mazizo de los cuerpos, de qualquiera suerte, o forma que vengan, *como en el lib. 4 se vera*" (my emphasis).

[2] Perez de Moya 1573, 6: "Y porque con mayor fundamento se pueda desto disputar, y dar razon: pondremos primero tres geeneros de principios, sobre que esta arte haze su fundamento. Que son *Diffiniciones, Peticiones, y Comunes sentencias, siguiendo en ello la orden que pone Euclides*" (my emphasis).

[3] Bullant 1562, I, 3: "il m'a semblé n'être hors de propos de pratiquer ce petit traité, contenant

of his 1561 treatise on gnomonics (which his practical geometry aimed to introduced from the second edition), artisans are described as (or related to) people who make use of the compass ("artisans et gens de compas").[1] Given the content of his treatise, which mainly deals with instrumental constructions of lines and figures, even if he does include a small part on surveying practices, it is clear here that he has in mind a constructive type of geometry, as described by Tartaglia as the second main part of practical geometry. Although Charles de Bovelles, in his 1547 *Geometrie practique* (or *Livre singulier et utile touchant l'art et pratique de Geometrie* in its first edition from 1542) did not clearly distinguish practical geometry from theoretical geometry, and did not as such identify theoretical geometry with Euclidean geometry, he offered a description of practical geometry similar to that of Bullant in his versified address to the reader. He then addressed his work to those "who seek the measures, and quantities of lines and figures, and of all bodies by the art of geometry, as well as several points and secrets of industry found most notable in this art" and "who bring their knowledge into effect", inviting them to use the geometer's instruments (the set-square, the straightedge and the compass), on which, he wrote, "depend the art and practice, as well as the profit of the whole of geometrical knowledge".[2] This description of practical geometry, although relatively vague, characterises it by its use of geometrical instruments, which includes the compass ("n'oublie pas | L'esquierre droict, la reigle & le compas"),[3] and by its emphasis on measurement ("les *mesures*, | Et *quantitez* des lignes & figures, | Et de touts corps"), as well as by its operative and productive character ("secrets

plusieurs règles et inventions géométriques, *sans parler de leurs spéculations et théoriques, ainsi qu'a fait Euclide*. Pour autant qu'elles *ne peuvent être si familières aux artisans*, comme elles sont aux gens doctes, et plus curieux".

[1] Bullant 1562, II, 4: "Et après avoir de long temps fait les épreuves d'iceux cadrans et horloges, ai bien osé mettre et réduire en notre vulgaire ce petit traité, *pour le profit et commodité des artisans et gens de compas*".

[2] Bovelles 1547, A1v: "Ami lecteur qui cherche les mesures, | Et quantitez des lignes & figures, | Et de touts corps, par art de Geometrie | Et plusieurs poincts & secrets d'industrie | Qui en c'est art sont trouvez plus notables, | Et pour les gens d'esperit profitables, | Qui leur scavoir redigent en effect: | Avoir te fault ce livre qui fut faict | Dedans Noyon par Charles de Bovelles, | Qui n'est jamais sans faire œuvres nouvelles. | Entens le donc, & si n'oublie pas / L'esquierre droict, la reigle & le compas: | Car de ces trois despend l'art & practique, | Et le profict du savoir Geometrique".

[3] This is supported then by an illustration of the three instruments placed underneath the address to the reader.

d'*industrie*"; "leur scavoir redigent en *effect*") and its usefulness ("de ces trois despend l'art & practique, | Et *le profict* du savoir Geometrique").[1] This description, furthermore, resonates which the sonnet Fine wrote for Bovelles's work,[2] in which it is written that: "All artisans and Mercurial people | Who desire to find new secrets | To measure need to have the practice [of geometry] | above all arts".[3]

In Jacques Peletier's *De l'usage de géométrie* (1573), the distinction between theoretical and practical geometry is situated rather between the "elementary geometry" (*geometrie elementaire*) of Euclid and the "geometry put into use" (*geometrie usagere*), which we may call here 'applied geometry'[4], of Archimedes, Apollonius and Ptolemy,[5] and of all those authors who "ingeniously conjoined the art with experience".[6] The type of geometry attributed to geometers such as

[1] On the aim and scope of Bovelles's *Geometrie practique*, see Oosterhoff 2014, Oosterhoff 2017 and Brioist 2021.

[2] Bovelles 1547, 2r: *Rhythmus circularis, Orontianus.* Fine assisted with the printing of Bovelles's work by engraving some of the figures and by supervising the printing of the text, as indicated in the Latin preface. Bovelles 1547, 2r: "(...) Orontius Regius Mathematicus (...) Duo protinus ingenuè spopondit. se quidem cum primis daturum operam, ut aereis tipis invulgata, plurimis esse visui: figurarum quoque quas ibidem frequentius inscripsi, futurum ligneis in tabellis pictorem. Necnon (quod praecipuum est) adversum mendas observaturum vigiles praeli excubias".

[3] "Tous artisans et gens mercuriaux, | Qui ont désir trouver secrets nouveaux, | De mesurer faut qu'aient la pratique | Sur tous les arts". Interestingly, this sonnet was taken up in Bullant's *Petit Traicté d'horologiographie et geometrie pratique* (Bullant 1562, I, 4), without the acknowledgement of Fine's authorship, as was the illustration of the set-square, straightedge and compass placed below it, which was originally included in Bovelles's treatise below his versified address to the reader (Bovelles 1547, A1v).

[4] The term 'applied' is used here in the sense that such knowledge would entail the use or application of general principles of geometry in order to solve concrete problems (in astronomy or mechanics or even in a more advanced geometry) without being reduced to a utilitarian scope, as would a type of geometrical knowledge properly aimed at craftsmen, as this term will be used below.

[5] It should be noted that the main criteria of distinction between Euclid, on the one hand, and Archimedes and Ptolemy, on the other, in Peletier's commentary on Euclid was the fact that the first arranged the propositions of geometry according to a properly ordered system, while the others did not astrain themselves to this rule. Peletier 1557, 12: "Ex utrisque Euclides Elementa Geometrica contexuit, ut operi vicissim subserviant, ordine quidem concinniori quàm ante illum quisquam: licet neque Archimedes, neque Ptolemaeus, neque ullus antiquorum se ordini astrinxerit".

[6] Peletier 1573, 3r: "Et de ma part je suis bien loing de l'opinion de ceux qui n'apellent *Geometrie* sinon celle *Elementaire, traitée par Euclide*, non pas *celle usagere d'Archimede, d'Apoloine, de Tole-*

Archimedes represents here a more advanced and more speculative type of "applied geometry" than that which is relevant to craftsmen and which is properly utilitarian, as was the major or higher computational practical geometry that was distinguished by Tartaglia from its minor or lower pendant in the *General trattato de numeri et misure*.[1]

1.5. The hybrid status of medieval practical geometry

In spite of this association between practical geometry and the use of geometry in the activities of surveyors and artisans, it is important to note that practical geometry, in the form it was given in the medieval Latin treatises by scholars such as Hugh of Saint-Victor, was in fact situated somewhere mid-way in its content, format, style and addressed audience between what we could call manuals of properly "applied" or "professional" geometry, which were aimed to be useful to specific kinds of readers among craftsmen (e.g. surveyors, merchants or barrel gaugers),[2] and the scholarly treatises on geometry, notably translations

mee & des autres auteurs excellens qui ont si ingenieusement conjoinct l'artifice avec l'experience" (my emphasis). It may be noted that the "applied geometry" of Archimedes, Apollonius and Ptolemy in the fields of mechanics, astronomy, cartography, gnomonics and in the resolution of complex geometrical problems, which Peletier distinguished then from Euclid's "elementary geometry", is much more advanced and by many aspects more theoretical than the teaching of surveying precepts that constituted the core of medieval *ars mensurandi*. Yet, it concords with the image of Archimede as associated with "fertility and invention", and as the model of engineers and architects, as demonstrated by E. Knobloch (in Knobloch 2005). Moreover, the types of problems dealt with in the *De usu geometriae* (see *infra*, §1.9 ff.) shows that Peletier considered the knowledge of the surveyors as somewhat connected to the more advanced applications of geometry in their aim and epistemological status.

[1] See *supra*.

[2] unlike the 'applied' geometry or *geometrie usagere* attributed to Archimedes, Apollonius and Ptolemy by Peletier, in his *De l'usage de geometrie*.

and commentaries of Euclid's *Elements*.[1] The former, such as the manuscripts of the medieval Italian abbacus tradition (which were primarily intended for the education of sons of merchants), agrimensorial treatises or even the few manuals written by master masons,[2] mostly corresponded to booklets written in the vernacular which contained a non-systematic collection of numerous problems that represented a variety of different cases on determinate and concrete quantities and set in specific contexts (real or fictitious).[3] Within these texts, the computation, measuring or construction procedures were generally taught directly through the examples as a set of direct instructions or "recipes" with little or no general expression and demonstration of the rule.[4] Euclid's *Elements*, by contrast, represented by essence (despite certain transformations made to the text throughout its medieval history[5]) an abstract consideration of the properties of geometrical objects, numbers, ratios and proportions, in which geometrical objects bear no specific dimensions and is dealt with separately from numbers and from any reference to instruments or to the material world, and within which the knowledge of magnitudes and of geometrical constructions is carried out demonstratively within propositions that are axiomatically ordered and derived from a set of abstract definitions and universal principles.

Latin medieval practical geometry treatises stand mid-way between these two textual traditions on the level of abstractness and generality. Even if they dealt with measuring techniques and artisanal practices, they were generally written in Latin by scholars, school masters and university professors and presented textual divisions and a level of generality more proper to scholarly treatises. They were also often bound in large mathematical compendia that would make them impractical to use on the field to solve concrete measuring prob-

1 Beaujouan 1975, Victor 1979, 42-53 and L'Huillier 1994. See also Morel 2020 for the case of *geometria subterranea*, or geometry of the mines, which can be considered as a particular genre of practical geometry.

2 Shelby 1972, Beaujouan 1975, Homann 1991, 7-18, Zaitsev 1999, Høyrup 2014.

3 On the distinction between real (or genuine) and fictitious problems in the manuscripts of the abbacus tradition (mostly concerned with commercial arithmetic, but which also contained problems of applied geometry), see Van Egmond (1980, 20).

4 On the content, style and scope of abbacus treatises, see Van Egmond (1980, 15-26) and Høyrup (2007, 27-44).

5 On the medieval Latin Euclidean tradition, more generally, see Murdoch 1971, Folkerts 1980 and Busard 2005, 1-40.

lems.[1] Hence, this textual tradition, which (as expressed by D. Raynaud[2]) held a marginal status with respect to both professional and scholarly geometry, dealt with a type of geometry that may itself be regarded as a hybrid of theory and practice in the sense that it offered a theoretical teaching on a practical form of knowledge.[3]

1.6. The development of practical geometry in the pre- and early modern era

Because of this intermediary status between applied or professional geometry and theoretical geometry and because of the lack of a canonical or standard textual reference (as in the medieval tradition which developed from Sacrobosco's *De Sphaera*),[4] the medieval Latin practical geometry tradition was highly multiform, presenting variations from one text to the other in the list of covered topics and on the types of problems that were dealt with or measuring techniques that were taught (notably between instrumental and computational procedures).[5] Throughout the Renaissance and the early modern period, the content, style, structure and aim of practical geometry treatises continued to evolve, coming to include an even more diverse group of topics, from the making and use of terrestrial and astronomical measuring instruments to the construction of complex curves, including also elements of algebra, trigonometry, engineering, astronomy, artificial perspective and certain Euclidean principles and propositions (directly drawn from editions of Euclid).[6] Certain practices which could be have been beforehand placed on the side of theoretical geom-

[1] L'Huillier 1994.
[2] Raynaud 2015, 19.
[3] Beaujouan 1975, Victor 1979, 42-53, L'Huillier 1994.
[4] On the *Sphaera* tradition in the middle ages and in the early modern period, see Thorndike 1949 and Valleriani (ed). 2020.
[5] On the diversity of medieval Latin practical geometry, see Victor 1979, 2, 17-18 and 29-30 and Moyon 2008, 144.
[6] On the multiformity of practical geometry and on the diversity of definitions of practical geometry in the early modern period, see Bennett 2002, Guyot and Métin 2004, Knobloch 2005, Raynaud 2015, 12-14 and Axworthy 2016, 250-262. See also Johnston 1996.

etry (notably the construction of geometrical figures on sand boards[1] or with the straightedge and compass when studying or teaching Euclid's propositions) would eventually be regarded as specifically belonging to a practical form of geometry,[2] being associated to a non-demonstrative and purely instrumental and concrete approach to the construction of figures, as represented notably by the constructive geometry used by artisans (as shown, for instance, in Durer's 1525 *underweysung der Messung*).[3]

This evolution had many factors, besides the preexisting multiformity of the medieval tradition, among which the invention of the printing press in the fifteenth century, which enabled a wider circulation of mathematical texts, the publication of newly discovered ancient geometrical treatises such as Pappus' *Mathematical Collection* and the creation of new pedagogical institutions and academies which gave a larger place to the teaching of mathematics, and to practical mathematics in particular within the mathematical curriculum (e.g. the Collège Royal in Paris, the Jesuit Collegio Romano or even Gresham College). It was also motivated by growing interests in the practical and technical applications of geometry to the resolution of concrete problems in the mixed sciences such as mechanics and astronomy (for instance, by professors of the university of Padua), to which should be added a more explicit demand for geometrical solutions in professional contexts,[4] all of which encouraged (and, for some of these factors, was in turn encouraged by) the publication of new works of practical geometry in Latin and in the vernacular.

As a category of pre- and early modern mathematical literature, practical

[1] See the preface to Johannes de Tinemue's commentary on Adelard's translation of Euclid, in Busard 2001, 32: "Instrumentum vero demonstratoris est radius et mensa cum pulvere".

[2] This appears, for example, through the fact that the instrumental performance of Euclid's construction was taught by Clavius, in his commentary on Euclid's *Elements* (first published in 1574), within sections entitled *Praxis*. On these sections, see *infra*, §1.6.

[3] On the difference between practical geometry as related to the applications of geometry in the artisanal practices and practical geometry as related to surveying practices (which was primarily called "practical geometry"), see Knobloch 2005. See also Shelby 1972 for the medieval tradition.

[4] See, for instance, the use of surveying for mining, which led to the development a specific kind of practical geometrical knowledge, called in Latin *geometria subterranea*. On this topic, see notably Morel 2020. T. Morel (in Morel 2017) showed how a greater demand for a proper teaching of Euclid's *Elements* adapted to the geometry of the mines was made explicit in the sixteenth century. On the use of Euclid in other professional contexts, notably for the training of artists, architects and engineers in the sixteenth century, see Camerota 2006 and Brioist 2009b.

geometry treatises certainly maintained a few essential characteristics, which (besides the acknowledgement that they belong to the genre of practical geometry in their title[1]) enabled to distinguish them from works dealing with a speculative type of geometrical teaching (as represented in particular by the versions of Euclid's *Elements* which claimed to be philologically accurate[2]). Among these characteristic features are first an emphasis on measurement procedures and on the construction and use of instruments to measure magnitudes or to construct or divide figures, and with this (notably the focus on measuring), a quasi-systematic numerical treatment of magnitudes and an appeal to computations, which implies an admission of approximation. In relation to this, is often expressed the explicit goal to propose procedures that are quick, easy, accessible and even pleasant to perform. To this, may be added the fact of placing the teaching of geometry within a socially determined context (by implicitly or explicitly pointing to pedagogical or professional uses of the taught procedures), an openess to methodological variety and/or novelty (by displaying multiple and sometimes entirely new modes of resolution of problems, either instrumental, computational or sensed-based) and, when offering a more general teaching on geometrical properties, the fact of privileging empirical over logical or rational modes of inference and demonstration.[3]

In other words, these treatises present geometry as a form of mathematical knowledge which is meant to be used, but also which is meant to be learned and constituted by the practitioner concretely and intellectually (in the mode

[1] This also the case of the treatises such as Johannes de Muris' *De arte mensurandi*, in which the title (or incipit) identifies the treatise as dealing with the object traditionally attributed to practical geometry, that is, the art of measuring. As pointed out by S. J. Victor (in Victor 1979, 15), Robert Kilwardby, in the *De ortu scientiarum* (Judy 1976, 29-30), used the names *ars mensurandi* and *ars mensoria* to designate a type of knowledge which corresponds in its content to the classical notion of practical geometry, as it deals with the measure of lengths, areas and volumes.

[2] This would, for instance, be represented by Zamberti's 1505 Latin translation from the Greek, as suggested by the title: "Euclidis megarensis philosophi platonici mathematicarum disciplinarum Ianitoris: Habent in hoc volumine quicunque ad mathematicam substantiam aspirant: elementorum libros XIII cum expositione Theonis insignis mathematici, *quibus multa quae deerant ex lectione graeca sumpta addita sub nec non plurima subversa & praepostere: voluta in Campani interpretatione: ordinata digesta & castigata sunt*" (my emphasis).

[3] D. Raynaud (in Raynaud 2015, 16-18) offers a description of the characteristics of practical geometry which is different by several aspects, though not in contradiction with the one presented here.

of a *quoad nos* form of knowledge), being susceptible thereby to be augmented and revised. This model thus clearly constrasts with the type of geometry represented by Euclid's *Elements*, conceived as an axiomatically-ordered system of propositions, since this work may appear as a finished and digested scientific synthesis of geometry, organised and formulated so as to filter out any explicit intention of the author or consideration of the reader (in the mode of a knowledge *secundum se*).[1]

Yet, from one treatise of practical geometry to another, the form and degree of importance which each of these characteristic features took, just as the list of topics dealt with in practical geometry treatises, could vary, and sometimes to a great extent. This diversity inherent to the pre- and early modern practical geometry tradition actually embodies the equivocity of the notion of *praxis*, from which was derived the very notions of practical knowledge and of practical geometry (or of practical mathematics more generally). The term *praxis*, which overall means action, had indeed in Greek Antiquity a technical, an epistemological, but also an ethical meaning, which explains the equivocity of the notion of practice and practical knowledge, even in contemporary English. This equivocity remained to a certain extent when applied to geometry in the pre- and early modern era, since the notion of *praxis* could be understood in various ways and according to different levels of abstraction when dealing with magnitudes, standing for the concrete operations of landmeasurers as much as for the resolution and demonstration of abstract geometrical problems, but also, less directly, to a form of *vita activa* in which the knowledge of geometry is valued for its utility in all aspects of human life, including moral and intellectual pursuits such as theology and philosophy.[2] Hence, throughout this

[1] On the difference between practical and theoretical knowledge according to the difference between knowledge *quoad nos* and *secundum se* in Renaissance philosophy of mathematics, see Higashi 2018, 113-120.

[2] This was, for instance, the case in Hugh of Saint-Victor's work, in which practical geometry is used in the estimation of the dimensions of Noah's ark. See Victor 1979, 32-34, and more generally on the speculative aims of practical knowledge in connection with the distinction between theory and practice in the middle ages, see Beaujouan 1975, Evans 1976 and Zaitsev 1999. We can also find this more philosophical and moral orientation of practical geometrical knowledge in Bovelles's *Geometrie practique*, which teaches, in chapter 7, as indicated by the title of the chapter, uses of geometry for the comprehension of the sound and harmony of bells, the pace of horses, carriages and charges, fountains, the struture of the cosmos on the dimension of the human body (*Septiesme*

evolution, the definition of the proper object, methods, finality and epistemological status of practical geometry, and its relation with theoretical geometry, was in constant reassessment, widening from within the spectrum delimited by applied or professional geometry, on one end, and by theoretical geometry, on the other.

1.7. The authors and addressed audience of sixteenth-century practical geometry treatises

It is important to note furthermore that, just as, in the middle ages, practical geometry treatises were written by scholars or professors, in the sixteenth century, when they were mostly published in print, such works were often written by university and college lecturers, as well as by humanists and court artisans, among whom several also published an edition, translation or commentary of Euclid's *Elements*.[1] This was the case of Luca Pacioli,[2] Oronce Fine,[3] Niccolò

Chapitre, Du son et accord des cloches, et des alleures des chevauls, chariots, & charges: des fontaines: & encyclie du monde: & de la dimension du corps humain). This chapter actually offers a greater number of topics such as the structure and construction of all sorts of mills, the motion of the physical elements, or the structure and number of the orifices of the human body.

[1] The very fact that the authors of practical geometry treatises were often the same as those who wrote an edition, a translation or a commentary on Euclid's *Elements*, or had a comparable social and professional status as the latter, would preclude the interpretation of the textual interactions between the sixteenth-century traditions of practical geometry and of Euclidean geometry as resulting from the existence, between their representatives, of a "trading zone", in the sense defined by Pamela Long (2011, chapter 4) on the basis of Peter Galison's analysis of the relations between the different "subcultures" of physics (Galison 1999).

[2] Pacioli, *Summa di arithmetica, geometria, proportione et proportionalità*. Venice: Paganino de Paganini, 1494 and *Euclidis megarensis philosophi acutissimi mathematicorumque omnium sine controuersia principis opera a Campano interprete fidissimo translata*. Venice: Paganinius, 1509.

[3] Fine, *In sex priores libros Geometricorum elementorum Euclidis Megarensis demonstrationes*. Paris:

Tartaglia,[1] Jacques Peletier,[2] Elie Vinet,[3] Pierre Forcadel,[4] Christoph Clavius[5] and Jean Errard.[6] By publishing a treatise of practical geometry, these authors may have wanted to display the extent of their geometrical expertise and gain thereby a greater visibility outside the university, attracting potential patrons. They may also have aimed to complete their pedagogical programme by offering their students a complementary teaching on geometry while offering an insight into the more concrete profit that could be gained from the study of geometry, besides knowledge and intellectual pleasure, or to offer an alternative means to learn geometry, that is, by familiarising oneself with the most fundamental geometrical notions at the same time as learning about the useful applications of geometry in a quicker, pleasant and more accessible manner. These practical geometry treatises, which were quite often written and published in the vernacular,[7] had, in this framework, a pedagogical, but also a recreational purposes, satisfying the curiosity of elite amateurs on the various uses and power of mathematics in various parts of concrete and intellectual life. These works were thus addressed, or at least would have been of interest

S. de Colines, 1536 and *Liber de geometria practica, sive de practicis longitudinum, planorum et solidorum*. Strasburg: G. Messerschmidt, 1544 (first published in the *Protomathesis*, Paris: G. Morrhe and Jean Pierre, 1532); *De re et praxi geometria libri III*. Paris, G. Gourbin, 1556.

[1] Tartaglia, *Euclide Megarense philosopho, solo introduttore delle scientie mathematice*. Venice: C. Troiano, 1543 and *General trattato di numero e misure*. Venice: 1556-1560.

[2] Peletier, *In Euclidis Elementa Geometrica Demonstrationum Libri sex*. Lyon: J. de Tournes and G. Gazeau, 1557 and *De usu geometriae*. Paris: Gilles Gourbin, 1572; *De l'usage de geometrie*. Paris: Gilles Gourbin, 1573.

[3] Vinet, *Definitiones Elementi Quincti et Sexti, Euclidis*. Bordeaux: S. Millanges, 1575 and *L'Arpenterie*. Bordeaux: S. Millanges, 1577.

[4] Forcadel, *Les Six premiers livres des Elements d'Euclide*. Paris: J. de Marnef et G. Cavellat, 1564; *Les septième, huictième et neufième livres des Élémens d'Euclide* . Paris: C. Périer, 1565 and *La Practique de la géométrie d'Oronce Finé, dauphinois, traduicte par Pierre Forcadel, du latin en français*. Paris: G. Gourbin, 1570.

[5] Clavius, *Euclidis Elementorum libri XV. Accessit XVI. De Solidorum Regularium comparatione*. Rome: Accoltus, 1574 and *Geometria practica*. Mainz: Hierat, 1604.

[6] Errard, *La Géométrie et practique générale d'icelle*. Paris: David le Clerc, 1594 and *Les six premiers Livres des Elémens d'Euclide* (...). Paris: Guillaume Auvray, 1598; *Les neuf premiers livres des elemens*. Paris: Guillaume Auvray, 1598.

[7] This is however by no means representative of a systematic pattern, as practical geometry treatises continued to be published in Latin throughout the sixteenth century, as shown notably by the case of Christoph Clavius's *Geometria practica* (1604).

to a mixed or hybrid audience, from scholars and university students to learned artisans and members of courts and government administrations.

1.8. The mutual influence of practical geometry and of Euclid's Elements in the sixteenth century

Partly because of a frequent overlap in authorship and context of diffusion, the printed practical geometry tradition started to converge more sensibly from the sixteenth century, in parts of its style and content, with the printed Euclidean geometrical tradition.[1] Indeed, a certain number of practical geometry treatises took up Euclidean material in a more recognisable and explicit manner and offered to teach Euclidean geometrical propositions in a practical way, for example, by translating the abstractly formulated constructions as instrumental procedures, by demonstrating the relations of figures through computations, or by displaying some of their concrete uses in everyday life. And certain characteristics proper to the practical treatment of Euclid's propositions, such as found in practical geometry treatises, could also be found within coeval adaptations, translations and commentaries on the *Elements*, as I will show in more detail in the second part of this article.

As such, in their printed form, both traditions, in the sixteenth century, mutually enriched each other in their content and approaches to geometry, making the boundary between practical geometry and theoretical geometry harder to define. In this framework, whereas the tradition of practical geometry treatises stemming from the Latin middle ages may be considered to offer a theoretical teaching of a properly practical or applied form of geometrical knowledge, translations and commentaries on the *Elements* that dealt with Euclidean concepts and propositions instrumentally, numerically or in a more empirical manner may be said to offer a practical teaching of theoretical geometry, both contributing to the hybridization of theoretical and practical geometry in the early modern era.

[1] The main previous studies which pointed to this phenomenon are Malet 2006 and 2012, Barany 2010, Barbin and Menghini 2014, Menghini 2015, Lee 2018.

1.9. The epistemological implications of the practical treatment of Euclid's Elements in the sixteenth century

While both types of geometrical texts may therefore be regarded as hybrids between theoretical and practical geometry in the premodern sense of these terms and categories, the fact that Euclid's *Elements* represented a canonical source, with an overall stable argumentative structure and number of principles, propositions and books, and also that it was presented in this context as the epitome of theoretical geometry and thus as essentially distinct from practical geometry, makes it a more obvious witness of the reshaping, by sixteenth-century mathematicians and professors of mathematics, of the content and form of geometrical knowledge in the direction of a less clear-cut distinction between the theoretical and practical approaches to geometry. Such an approach would promote a more hands-on, accessible, useful, socially relevant and innovative approach to geometry and extend thereby the scope of scholarly geometry, in its objects and methods, beyond the Euclidean framework.

This more practical reading of Euclid's text, which is to be found both in Latin and in vernacular translations and commentaries of the *Elements*, could, in the eye of Renaissance mathematicians and humanists, be partly justified by the common admission (based on the reports of ancient authors, such as Proclus) that ancient Greek geometry, and thus of Euclidean geometry, held its historical origin in the practice of Egyptian measurers.[1] Moreover, it could be based on the acknowledgment that a part of the content of practical geometry was

[1] Bovelles 1547, 3r-v: "L'art de Geometrie selon les anciennes histoires, fut jadis trouvé en Aegypte, a cause de la riviere du Nil"; Perez de Moya 1568, 1: "Geometria, aun que puede significar mas cosas, propriamente es Arte de medir la tierra, inventada de los Egypcianos (como refiere Strabon) por la inundaciones que el Nilo hazia"; Merliers 1575, 1r: "Geometrie est l'art de mesurer, lequel selon les histoires anciennes a prins son origine & commencement des Egyptiens pour la necessité des limites & bornes de leurs terres, lesquelles le Nil au temps de son desbordement couvroit de limon, en

ultimately based on Euclidean geometry (albeit not through a direct reading of the *Elements*), and perhaps also on contemporary classroom pedagogical practices as on the fact that certain medieval translations of the *Elements*, such as the commentaries based on Adelard's translation from the Arabic by Johannes de Tinemue and Campanus, already contained certain practical features, such as the mention of geometrical instruments (i.e. the compass) in the first two propositions of Book I.

Even so, the fact that Euclid's text could be dealt with in a practical manner was not self-evident in the sixteenth century. Indeed, even for the mathematicians who acknowledged geometry (in general) as having originated in the practice of measurers (such as Bovelles, Merliers and Clavius), the geometrical knowledge represented by Euclid's *Elements* was presented as having progressed beyond the more basic geometrical knowledge of surveyors and as having evolved into a properly universal and necessary science,[1] which deals with

sorte qu'apres iceluy on ne les pouvoit plus recognoistre, qui estoit cause de confusion & de trouble: pour à quoy remedier fut ordonné par les Roys d'Egypte, que par les prestres qui estoyent oysifz & sans payer tribut, fut trouvé quelque art de si bien mesurer & borner les terres, que par l'annuel desbordement du Nil ne fussent plus confondues ny troublées"; Ramus 1569, *Geometria libri XXVII*, 2: "geometria hunc in modum definita sit ars bene metiendi. Nomen autem re ominata levius est. Geometria enim dicitur tanquam terrae tantum dimetiendae ars quaedam sit, quod nomen videtur in Aegypto primum factum esse: ubi ad terminos agorrum Nili inundationibus obrutos restituendum primum geometria adhibita sit"; Clavius 1611-1612, 4: "Geometria vero, aucotre Proclo, ab Aegyptijs reperta est, ortumque habuit ab agrorum emensione. Cum enum anniversaria Nili inundatio agrorum terminos, ac limites ita confunderet, vastaretque, ut nemo agrum dignoscere posset suum, coeperunt Aegyptij animos ad rationem mensurandorum agrorum applicare, ut hoc modo cuilibet, quod suum erat, redderetur. Quae quidem ratio agros metiendi, quanquam tunc temporis adhuc rudis admodum fuerit, ac impolita, ab ipso tamen officio Geometria est appellata. γεωμετρέομαι *enim, sive* γεωμετρέω idem significat quod, terram metior". On this historical account of the origin of geometry and its reception in the Renaissance, see Goulding 2010, xix–xx and 1-14.

[1] Bovelles 1547, 3r-v: "Apres les prebstres d'Aegypte, plusieurs autres gens scavants & de grand engin, ont adiousté & fort augmenteé la science de Geometrie, comme Pythagoras, Archimedes, Euclides, duquel le livre est a present imprimé, & par tout divulgué"; Merliers 1575, 1r: "Apres, les prestres d'Egypte, plusieurs ont augmenté ledit art, & encores tous les jours par le labeur & speculation de gens Doctes croist & enrichit: Car il n'y-a art si parfaict, que chacun jour par nouvelle invention ne se puisse bien augmenter & mettre à plus grande perfection"; Clavius 1574, sig. b1r: "Immo vero singulas [disciplinas Mathematicas] nequaquam summam adeptas esse perfectionem statim ab initio, sed paulatim eas ab imperfectis ad perfectiora processisse, memoriae quoque proditum est. (...) Caeterum paulatim deinde Geometria caepta est expoliri, & non contenta suis finibus, sese ad corpora etiam caelestia dimetienda convertis, tradititque principia universae Astronomiae, Perspecti-

the properties of magnitudes *per se* (or with intelligible, as opposed to sensible, magnitudes) and in a manner that is demonstrative and rational (as opposed to empirical), and which furthermore managed to be regarded as a model for other sciences in this respect.

For this reason, John Dee considered that the Greek designation *geometria* (γεωμετρία), which literally means 'measure of the earth' (coming from γῆ, the earth, and μετρέω, to measure), would be improper for this science and should rather be called *Megetologia* or *Megethica* (i.e. 'discourse on magnitude' or '[doctrine] concerning magnitudes', from μέγεθος, 'size').[1] This conception of (theoretical) geometry as a purely speculative and demonstrative science, dealing with intelligible magnitudes, thus led certain commentators of Euclid, such as Peletier and François de Foix-Candale, to filter out of Euclid's *Elements* certain modes of demonstration which they judged too empirical or mechanical to be admitted in geometry, such as the use of superposition to demonstrate the congruence of figures.[2] This conception was comforted by the circulation of the

vae, Cosmographiae, & alijs disciplinis quam plurimis, quae ex ipsa, veluti radices dependent. Hanc Thales Milesius ex Aegypto in Graeciam primus transtulisse fertur: Deinde eam insignes Philosophi, ac Mathematici plurimis, acutissimisque demonstrationibus locupletarunt, atque exornarunt". This was also the discourse held by Proclus, who represented a key source of sixteenth-century scholars on the early history of Greek geometry. Proclus 1873, 64; transl. Morrow 1992, 52: "geometry was first discovered among the Egyptians and originated in the remeasuring of their lands. This was necessary for them because the Nile overflows and obliterates the boundary lines between their properties. It is not surprising that the discovery of this and the other sciences had its origin in necessity, since everything in the world of generation proceeds from imperfection to perfection. Thus they would naturally pass from sense-perception to calculation and from calculation to reason".

1 Dee, in Billingsley (1570, sig. a2r-v): "This Science of *Magnitude*, his properties, conditions, and appertenances: commonly, now is, and from the beginnyng, hath of all Philosophers, ben called *Geometrie*. But, veryly, with a name to base and scant, for a Science of such dignitie and amplenes. (…) The people then, by this art pleasured, and greatly relieved, in their landes just measuring: & other Philosophers, writing Rules for land measuring: betwene them both, thus, confirmed the name of *Geometria*, that is, (according to the very etimologie of the word) Land measuring. (…) An other name, therefore, must nedes be had, for our Mathematicall Science of Magnitudes: which regardeth neither clod, nor turff: niether hill, nor dale: neither earth nor heaven: but is absolute *Megethologia*: not creping on ground, and daddeling the eye, with pole perche, rod or lyne: but liftyng the hart above the heavens, by indivisible lines, and immortall beames meteth with the reflexions, of the light incomprehensible: and so procureth Joye, and perfection unspeakable. Of which true use of our *Megethica*, or *Megethologia*, *Divine Plato* seemed to have good taste, and judgement: and (by the name of *Geometrie*) so noted it: and warned his Scholers therof".

2 Peletier, 1557, Prop. I.4, 16: "*Figuras Figuris superponere, Mechanicum quippiam esse: intelligere*

Greek text of Euclid's *Elements* in manuscript from the end of the fifteenth century and in print from 1533, which offered a version of Euclid's text that was sensibly different from some of the Latin versions translated from the Arabic that circulated in the middle ages, including that commented by Campanus and which corresponds to the very first version of Euclid's *Elements* to come into print in 1482. The humanist Bartolomeo Zamberti thus produced an entirely new translation of the *Elements* from the Greek which intended to restore the purity of Euclid's text since it had been, according to him, corrupted by the previous translation from the Arabic circulated by Campanus' commentary.[1] This version notably left aside (as per the Greek text) the mention of the compass included in Campanus' text.

Hence, the fact for certain sixteenth-century translations and commentaries on Euclid's *Elements* to take up features that were characteristics of practical geometry, and this to a much greater extent and in a much more explicit manner than in the altered versions transmitted by the medieval tradition,[2] contrasts

verò, id demum esse Mathematicum". See also Peletier, 1557, Prop. I.4, 15: "Haec est vulgata omnium Interpretum Demonstratio, si modò haec Demonstratio dici debeat. Nam si linearum figurarumque superpositiones in probationem recipiamus, tota ferè Geometria huiusmodi applicationibus erit referta: vixque ulla occurret Propositio, quae hac ratione non possit probari. Secunda enim iam indè ac tertia, quas modò demonstravimus, sic probari poterant. Nam si ad datum punctum, linea datae lineae aequalis ducenda sit: illicò translata linea ad ipsum punctum, absolutum erit negotium: Applicatio verò quanvis superpositione sit tolerabilior, tamen in Geometria repudiatur: immò ne lineam quidem transportare licet, ut secundum ipsius magnitudinem, Circulum describamus: quin prius aequalis linea ducta sit. Alioqui secunda prorsus vacaret. Tum si à maiori linea, minor sit abscindenda: quid aliud quàm maiori minorem superponemus, ut quod superat resecemus? *Sed hoc quàm sit à Geometriae dignitate alienum, eorum iudicio relinquo qui Demonstrationis vim & energiam animo concipiunt*" and Foix-Candale 1566, fol. 5v, Prop. I.4: "Alteram demonstrationem huic quartae exhibere cogimur, ne praebeatur aditus, quo *ulla mechanicorum usuum instrumenta in demonstrationes incidant.* Nam Campanus ac Theon hanc demonstrantes, triangulum triangulo superponunt, angulumque angulo, sive latus lateri, demonstrationem *potius instrumento palpantes, quàm ratione firmantes*: quod tanquam *prorsus alienum à vero disciplinarum cultu reijcientes*, aliam demonstrationem absque figurae, anguli seu lineae transpositione, protulimus ratione elucidatam". Cf. Foix-Candale 1566, 6v, Prop. I.8: "Huius alteram demonstrationis partem resecavimus eò quòd trianguli transpositione uteretur, quod quidem *mœchanicum spectat negotium à vera mathesi alienum*, posita anguli qui ad z hypothesi ex quarta huius sumpta" (my emphasis). On this issue, see Mancosu 1996, 29-31, Loget 2000, 171-177, Palmieri 2009, 474-476, and Axworthy 2018.

[1] Høyrup 2019.

[2] For instance, the Latin versions of Euclid's *Elements* circulating in the middle ages only rarely

with this image of Euclidean geometry as a purely rational science of intelligible magnitudes, in other words as a properly theoretical form of geometry, contributing therefore to transform the conception of the science of geometry such as represented by Euclid's *Elements*.

1.10. Justifications and motivations for the practical treatment of Euclid's Elements

Looking again at the introductions of sixteenth-century practical geometry treatises in which Euclidean geometry was held as representative of theoretical geometry, we do not only find a clear distinction between the two part of geometry, but also an assertion of their connection, in the sense that Euclid's *Elements* would correspond to the necessary introduction to practical geometry, enabling to demonstrate the causes on which the procedures taught by practical geometry are founded. Practical geometry would, in turn, show the profit that may be obtained from the study of geometry in general, or of Euclid's theoretical geometry more specifically, by teaching its various uses and utility in everyday life. This is clearly asserted by Fine in the introduction of his *Geometria practica* (or second book of his 1532 *Geometria*), in which he presented Euclid's geometry as the theoretical rudiments that one must learn in order to obtain the fruit that is to be reaped from the study of geometry, that is, the knowledge of the measures of lines, surfaces and bodies and of the use of geometrical instruments.[1]

referred to instruments (this is the case in the versions based on Adelard's translation, i.e. Robert de Chester's redaction or Adelard II in Prop. I.1; Johannes de Tinemue's commentary or Adelard III in Prop. I.1-2 and in the commentary of Campanus, Prop. I.1-2). They also practically never made references to concrete applications (an exception is found in Johannes de Tinemue's commentary or Adelard III, Prop. X.8 (= X.12), as noted in Murdoch 1968 and confirmed in his response to G. Beaujouan, in Beaujouan 1975, 480). Even if specific numbers (instead of lines representing numbers) were used in the context of Book V, which deals with the theory of ratios and proportions applied to magnitudes (to show that the properties of ratios, even if treated with separately by Euclid for numbers and for magnitudes, was common to both), and in the arithmetical books, numerical values are not directly applied to geometrical lines and figures, nor are computational methods used to measure and compare magnitudes.

[1] Fine 1532, 64r: "Duo sunt, optime lector, quæ in omni disciplina, studiosis omnibus solent esse non iniucunda. unum est, facilis in disciplinam introductio: qua & via doctrinæ, & sensus eius-

The fact that the knowledge of the theoretical principles of geometry is necessary to correctly apprehend the precepts of practical geometry was also held by Reisch[1] and Chauvet.[2] Even in those texts that did not clearly mention the theoretical pendant of practical geometry, such as those of Jean de Merliers and Giovanni Pomodoro and Christoph Clavius, Euclid's *Elements* was indirectly or implicitly acknowledged as the theoretical foundation of practical geometry. This was indicated in the title of Pomodoro's book[3] and, in the texts of Merliers and Clavius, through a frequent or quasi-systematic reference to the relevant propositions of Euclid,[4] as in Fine's practical geometry.[5] A similar system was

dem universus aperitur. Reliquum esse videtur, collectus ex ipsa disciplina fructus, susceptorum laborum compensator gratissimus. Præmissis itaque generalibus ipsius *Geometriæ rudimentis, ad elementorum Euclidis, & succedentium nostrorum operum intelligentiam isagogicis*: *consequens* nobis visum fuit, *universam Geometriæ subnectere praxim, hoc est, linearum, superficierum, & corporum, ex demonstratis Euclidis elementis, ostendere mensuram*". See also the title of Fine's book on practical geometry: "Liber secundus Geometriæ, *de practicis longitudinum, planorum & solidorum*, hoc est, linearum, superficierum, & corporum mensionibus, alijsve mechanicis, *ex demonstratis Euclidis elementis corolarius*: ubi et de quadrato geometrico, et virgis seu baculis mensorijs" (my emphasis).

[1] Reisch 1504, r2r: "nam tibi in prioribus geometriae generaliora principia, paucasque eiusdem theorias proposui. Iam vero quia te vigilantissimum, eiusque percupidum sentio: de metiendi modo pauca subiungere aenitar".

[2] Chauvet 1585, 1r: "pour autant que *la Theorique* est suffisamment traictée & *demonstrée aux Elemens d'Euclide*, nous en ferons un tacet, & renvoirons le nouveau aprentif à *iceux principes, sans lesquels il ne peult parvenir à la vraye cognoissance de la Geometrie Pratique*" (my emphasis).

[3] Pomodoro 1599, title page: *Geometria prattica, tratta dagl'Elementi d'Euclide et altri Auttori (...) Nella quale si vede in 50 Tavole di Rame scolpito tutto quello che ad un buon Giometra s'appartienne di sapere et porre in uso. Opera non meno uttile che necessaria, a' Misuratori di terreni, di fabriche, et altri simmili, ma' in'oltre ancora a' Geografi, Cosmografi, Architetti Civili, et Milittari, a' Bombardieri, Soldati privati, a' Capitani, Maestri di Campo, et a' qual si Voglia altra persona Virtuosa.*

[4] See, for example, Merliers 1575, 3v: "Pour sçavoir le contenu d'iceluy, cest à dire combien il contient de perches quarrées, ou de mesures quarrées: multiplions 8 par 6 ou 6 par 8: car il en sera produict un mesme nombre *par la seziéme proposition du septiéme livre des Elemens d'Euclide, qui dict*: Si deux nombres se multiplient l'un l'autre, les engendrez d'iceux seront egaux nous aurons 48 perches quarrées pour le contenu dudict Parallelogramme. *Ce moyen est prins de la trente-siziéme proposition du premier livre des Elemens d'Euclide, qui dict*: Que tous Parallelogrammes, qui sont en bases egales & en mesmes Paralleles sont egaux ensembles". and Clavius 1604, 65-66: "Ex vertice montis, vel turris per duas stationes in aliqua hasta erecta, vel in duabus fenestris turris, quarum una supra aliam existat, factas, è quibus signum aliquod in Horizonte videri possit, altitudinem ipsius montis, aut turris metiri. (...) Quia vero *ex schol. propositionis 4 lib. 6 Euclid.* est", or 175-177: "De area triangulorum (...) *ex coroll. prop. 4 lib. 6 Euclid*" (my emphasis).

[5] Fine 1532, 65r-v: "Qualiter in plana terrestri superficie iacentes lineae rectae, per ipsum quadra-

followed in the *Géométrie et practique générale d'icelle* of Errard, who did not clearly distinguish nor define practical geometry and theoretical geometry, but who asserted in his address to the reader the relevance of certain Euclidean demonstrations for his teaching of practical geometry.[1]

In Tartaglia, Perez de Moya and Peletier, the connection between theoretical (Euclidean) geometry and practical geometry took the form of a more systemic interdependence in the sense that, for Perez de Moya, theoretical or speculative geometry (which is explicitly identified with Euclidean geometry) would "consider quantity and proportion through a speculation of the mind" "in order to find the cause of the effects of practical geometry" and practical geometry would aim to "put into effect or implementing the reasons on which the mind reflects in theoretical geometry", as if they properly shared a common aim and could not be considered without the other.[2] As for Peletier, he wrote in his preface to *De l'usage de geometrie* (1573), when talking about theory and practice in general (for which geometry is then taken as a particular example), that neither theory, nor practice can be brought to perfection without the other: "they are two parts that are so indebted to each other that nobody could understand any art in its perfection without their mutual agreement and relation".[3] When

tum metiantiur geometricum. (...) Quoniam angulus AEB, aequus est angulo DAF, nempe alterno, *per 9 primi elementorum Euclidis* (...) & similis sunt rationis, quae aequalibus angulis latera subtenduntur, *per quartam sexti elementorum eiusdem Euclidis*". (My emphasis).

[1] Errard 1594, A2v: "*J'y ay entrelassé quelques demonstrations des elements d'Euclide* (comme le corollaire I du chapitre 3 du deuxieme livrez: le Corollaire 4 du chapitre suyvant, & quelques autres, *que j'ay estimé necessaires, pour la pratique parfaicte de la Geometrie*) la duplication du cube & division de la sphere, avec ce qui en depend, y sont demonstrees (combien que la pratique en soit mechanique) autant facillement & exactement qu'il s'est peu faire jusques à present" (my emphasis). See Métin 2016, I, 236-237.

[2] Perez de Moya 1573, 5: "La Theorica, ò Speculativa es aquella, que por hallar la causa de los effectos de la Practica, considera la quantidad, y proporcion con una especulacion del entendimiento, de lo qual trato Euclides compendiosa y cumplidamente (...). La Practica trata, de poner en effecto, ò en obra las razones que el entendimiento en la Theorica Speculo". Cf. Tartaglia 1556, III, 1r: "Delle specie della Geometria. Le specie principali della geometria sono due, delle quali l'una è detta theorica, & l'altra pratica. La theorica è quella che per investigare le propinque cause de gli effetti di quella, considera, & guarda le quantita, le proportioni, & le misure di quelle, con una speculatione di mente, & di questa abondantemente ne parla, & tratta Euclide Megarense in dodici libri".

[3] Peletier 1573, 2r: "Entre les hommes de sçavoir & d'experience, Monseigneur, a esté communement douté, laquelle doit estre preferée selon l'ordre de nature & de dignité, ou la Theorique, ou la Practique. De laquelle controverse est difficile de trouver l'issue. Et n'estant icy le lieu de raisonner

illustrating this through the example of geometry, he stated that, within it, one may find both demonstrations and operations, for which the *geometrie usagere* of Archimedes, Apollonius and Archimedes should be, as much as Euclid's *geometrie elementaire*, properly considered as belonging to geometry.

> But so as to not enter too deeply into this too general discourse, I would only take as an example of this argument our geometry, which brings an infinite pleasure through the contemplation of such a beautiful arrangement, [a contemplation that is] so well endowed with infallible, necessary and impugnable causes and reasons, and which brings, on the other hand, the greatest convenience in the practice and handling (*exercice et maniement*). (...) And, as for me, I am far from the opinion of those who only call geometry that which is elementary, and with which Euclid deals, and not the geometry put into use (*celle usagere*) of Archimedes, Apollonius, Ptolemy and of the other excellent authors who have so ingeniously conjoined the art with experience.[1]

Where Peletier goes beyond Trtaglia and Perez de Moya (and *a fortiori* beyond Reisch, Fine, Chauvet and all the other authors mentioned here) in his description of the connection between theoretical and practical geometry is that he presented both as possessing a speculative and an operative or practical part. Hence, when describing geometry in general, he implies this by saying that all human endeavors are governed by "measure and proportion",[2] that is, the main objects that geometry investigates in a demonstrative manner, and that in all speculative contemplation there is an operative part and an intention of application.

d'une part & d'autre, il me suffira de dire que *ce sont deux parties qui se rendent tel devoir ensemble, qu'on ne sçauroit entendre un artifice en sa perfection, sans la convenance & rapport de l'une avec l'autre*" (my emphasis).

[1] Peletier 1573, 2v-3r: "Mais pour n'entrer point si avant en ce discours trop universel, j'employeray pour exemple de cet argument nostre seule Geometrie, laquelle apporte un infiny plaisir en la contemplation d'une si belle ordonnance & si bien garnie de causes & raisons infaillibles, necessaires, irrepugnables: & d'autre part une commodité amplissime en l'exercice & maniment. (...) Et de ma part je suis bien loing de l'opinion de ceux qui n'apellent Geometrie sinon celle Elementaire, traitée par Euclide, non pas celle usagere d'Archimede, d'Apoloine, de Tolemee & des autres auteurs excelles qui ont si ingenieusement conjoinct l'artifice avec l'experience".

[2] This assertion was already set forth by Peletier in his commentary on Euclid's *Elements*. Peletier 1557, sig. 4r-v: "Nihil enim in rebus humanis ferè aliud est quod expediat aut iuvet, praeter ordinem & proportionem: id est, in omnibus moderationem. ubique igitur latet vis quaedam Geometriae".

For there is no endeavor (*negoce*), great or small, which is not either openly or secretly sustained and as animated by measure and proportion, the two essential parts of geometry and, on the other hand, there is no knowledge (*intelligence*) so abstruse that it does not take on and is not endowed with an intention of effect and application (*intention d'effect & d'execution*).[1]

For this reason, the intention of Peletier's book would be to conjoin theory and practice.

For these reasons, (…) although I have entitled this book "On the use of geometry", I could not, or should not, do otherwise than first provide the purely theoretical principles before dealing with the matter indicated by the title, *as it is my main goal to make these two parts conjoined with one another.*[2]

Although Peletier says here that the theoretical part of geometry is dealt with in the first section of the book (which contains a list of Euclidean definitions taken from Book I), and that the practical part of geometry will be presented afterwards, the content of his treatise actually shows that, within the very part dedicated to practical geometry, Peletier attempted to bring together theory and practice.

This would, in a way, correspond to the manner in which Archimedes, Apollonius and Ptolemy, according to Peletier, would have "ingeniously conjoined the art with experience" in their geometrical work. In Peletier's *De usu geometriae* (or *De l'usage de géométrie*), this first takes the form of a properly practical treatment of Euclidean problems (which mainly corresponds to the fact of leaving aside Euclid's demonstrations and of teaching how to perform the construction instrumentally or through computations),[3] to which are added elements proper to the more traditional notion of practical geometry as an art of

[1] Peletier 1573, 2v-3r: "Car il n'y a negoce, pour grand ou petit qu'il soit, qui ne se trouve ou apertement ou couvertement entretenu & comme animé de mesure & de proportion, deux parties essentielles de Geometrie: de l'autre part, n'y a intelligence si abstruse, qui ne soit revetue & incorporée d'une intention d'effect & d'execution".

[2] Peletier 1573, 3r: "Pour ces causes, Monseigneur, combien que j'aie intitulé ce Livre, de l'usage de Geometrie, si non pouvoy-je, ny devoy faire autrement que je ne premisse les Principes purement Theoriques, avant qu'entrer en matiere de ce que le Titre propose: *estant mon principal but de rendre ces deux parties conjointes ensemble*" (my emphasis).

[3] An example of this is presented *infra*, §2.4.

measuring by instrumental means (teaching the mode of composition and use of a surveying instrument of Peletier's design)[1] and the resolution of various geometrical problems (such as the construction of an asymptotic curve or the instrumental construction of two mean proportionals, necessary to the resolution of the duplication of the cube).[2] The fact of only taking up problems from Euclid's *Elements*, as opposed to theorems, shows that Peletier acknowledged the intrinsically practical character of *problemata*, even if these were dealt with demonstratively in the *Elements*, unlike in his *De usu* or *De l'usage*. Hence, when introducing his treatment of Euclidean problems in the latter work, Peletier wrote that the doctrine provided then is "half-mechanical and half-speculative", whose mechanical aspect is represented by the "use of the compass, the ruler, and the other instruments which conform themselves to the practice of geometry".[3] As I will briefly show later, one finds a similar treatment of Euclidean problems in Perez de Moya's *Geometria practica y speculativa*, in Digges's *Pantometria* and in Robert Recorde's *Pathway to Knowledge*.[4]

A discourse of the sort would certainly justify the fact of introducing a more practical treatment of Euclid's propositions in the context of a commentary on the *Elements*, but it was not properly materialised in Peletier's commentary on Euclid, nor even in that of Fine[5] (among the authors of practical geometry trea-

[1] Peletier 1572, Probl. 27, 31: "De ratione metiendi intervalla & altitudines, unica statione & in uno pede"; Peletier 1573, 47: "Mesurer les distances & hauteurs, par une seule station".

[2] Peletier 1572, Probl. 28, 38: "Datae lineae rectae lineam ascribere, quae ad ipsam continuè accedat, nunquam tamen cum ipsa concurrat, etiam infinitè protracta"; Peletier 1573, 58: "A une ligne droitte donnee accommoder une autre ligne, laquele approche tousjours d'icelle droitte, sans jamais se pouvoir conjoindre à elle, fussent elles infiniment alongees". Peletier 1572, Probl. 29, 40: "Inter duas rectas lineas datas, duas lineas continuè proportionaleis mechanicè reperire"; Peletier 1573, 61: "Entre deux lignes donnees trouver mechaniquement deux lignes continuellement proportionales".

[3] Peletier 1573, 15: "nous donnons icy une *doctrine moitié mechanique & moitié speculative*: nous servans icy de l'usage du Compas, de la Regle, & des autres Instrumens qui s'accommodent à la prattique de Geometrie" (my emphasis). In the Latin version, this teaching is only described as mechanical. Peletier 1572, 7: "Quo fit, *ut hic etiam mechanica doceamus*: usum scilicet Circini, Regulae, aliorumque instrumentorum quae ad opus Geometricum accommodari solent".

[4] In this work, which offers an English adaptation of Euclid's first four books of the *Elements*, the definitions, mostly taken from Book I, and the problems are dealt with together in the first book, and the postulates, common notions and theorems are dealt with in the second book. For the comparison mentioned in the text, see *infra*, Table 2.

[5] Indeed, apart from certain rare occurrences or certain aspects that were not practical *per se*

tises who asserted the interdependence of practical geometry and of Euclidean geometry[1]), even if these two authors did point, in their exposition of the *Elements*, to the fact that Euclid's geometry itself contained a practical part in the form of problems, in the sense that the problems would teach how to *do* something (e.g. construct or divide a figure or find a requested geometrical object), as opposed to theorems, whose aim is to demonstrate, and thus *know*, universal facts concerning geometrical objects.[2]

Among the authors that did present a more clearly practical approach to Euclid's *Elements*, the inherently practical character of problems was asserted by Tartaglia, in relation to his distinction between practical and theoretical geometry.[3] And, in this context, problems are described as practical in a much more concrete manner, that is, as teaching processes useful to artisans.

(though they become so when combined to other aspects), Fine and Peletier seemed overall intent, in their commentaries on Euclid, to maintain the abstract and rational character of Euclid's principles and propositions. This notably appears, in Peletier's commentary, by his rejection of superposition on account of its allegedly empirical character.

[1] It is furthermore important to note that Fine was himself, and from the beginning of his adult life, interested and engaged in practical applications of geometry as an intrument-maker, cartographer and engraver. On Fine's activities as an intrument-maker, cartographer and engraver, see notably Brioist 2009c, Eagleton 2009, Turner 2009 and Pantin 2013.

[2] Fine 1536, 10: "Ex his itaque sane quam intellectis principijs, colliguntur *problemata: hoc est, ambiguæ propositiones, sciscitationésve, practicas figurarum affectiones discutientes*: & Theoremata, id est, speculativæ propositiones, præceptionis utcunque participes, quæ singulis accidunt figuris sola inspectione diiudicantes". and Peletier 1557, 12: "Problemata, ortus Figurarum comprehendunt, sectiones, additamenta: eaque omnia in arte, quae facienda proponuntur. Atque, ut in Philosophia, Problemata dicuntur dubia quaedam quae nobis examinanda & solvenda proponimus: sic in Geometricis, Problemata vocamus constructiones ex arte depromptas: à quibus speculationes oriuntur, seu Theoremata: nempè quae factas Figuras comitantur, proprietates & affectiones: quaeque scientiae ipsi inhaerent & ipsam efficiunt. Nam in assertiones consistunt praeceptorum sicut Problemata in constitutione Figurarum. Ad summam, *Problemata materiam quandam praxinque artis referunt*: Theoremata, formam, scientiaeque meditationem" (my emphasis).

[3] Tartaglia 1543, 3v: "Anchora inanzi che piu oltra procediamo bisogna notar qualmente la scientia di Geometria, & di Arithmetica se divide in due specie, una del lequal (come fu detto in principio) é detta Theorica, cioe, speculativa, over contemplativa: l'altra è detta prattica, cioe, attiva, over operativa. (…) Euclide adonque per darci il fondamento d'una e dell' altra specie, ci ha descritto nell'Opra sua di *due specie propositioni*, l'una del lequal ce introduce nella theorica, cioe, nella parte speculativa: *& l'altra, ci conduce alla prattica, cioe, nella parte operativa.* Le propositioni adonque che ci conducono nella speculativa Grecamente si dicono Theoreme: *& quelle che ci guidano alla operativa si dicono Probleme*" (my emphasis).

And from these problems one learns the manner and way to draw, describe, inscribe, circumscribe, divide and form not only all properties of the plane figure with all of the accidental conditions that may occur in Painting, Perspective, Iconography, Chorography, Scenography, Geography, and Cosmography, but also all the various properties of the solid body with all of the subtle and accidental conditions that may occur, not only in Orthography, Sculpture, and Architecture, but in any other ingenious operation that depends from them, as one will be able to clearly see along the way.[1]

The fact that, in this passage, problems are said to teach all sorts of geometrical procedures relevant to artisans would be enough to justify the fact of dealing with Euclid's *Elements*, notably Euclid's problems, according to a practical approach, all the more as Tartaglia's Euclid corresponds to the first vernacular translation of the *Elements* and was as such intended to make Euclid's treatise more accessible to a more common audience.[2]

Yet, despite the fact that he translated Euclid in Italian and in spite of the practical features he introduced in his exposition of the *Elements* (some of which will be analysed in the next pages), Tartaglia did not so much teach the artisanal applications of Euclid's problems and he made sure, more generally, to clearly distinguish the more intellectual and scientific approach to geometrical propositions from their empirical and non-scientific treatment,[3] even more than other

[1] Tartaglia 1543, 3v: "Et da dette Probleme si apprende il modo & la via di dissegnar, discrivere, inscrivere, circonscriver, divider, e formar non solamente ogni qualita di figura superficiale con tutte quelle accidental conditioni che occorrer possano in Pittura, Prospettiva, Ichnographia, Corographia, Scenographia, Geographia, & Cosmographia, ma anchora ogni varia qualita di corpo solido con tutte quelle sottil & accidental conditioni che occorrer posiano, non solamente nella Orthographia, Scultura, & Architettura, ma in ogni altra ingeniosa operatione da queste dependente, come procedendo manifestamente si potra vedere".

[2] Tartaglia 1543, 3r: "Onde fra me pensando alla grandissima utilita che di queste due discipline ne consegue percoloro che le sanno secondo li debiti bisogni allo intelletto accommodare, accio che quelle tornino nel pristino stato, & che l'Opra dello ingeniosissimo Euclide sia riconosciuta, non solamente ho vogliuto durar questa fatica di riassettarla & integrarla secondo le due tradottioni, ma etiam *per commune utilita dal latino in volgar tradurla, & dilucidarla con espositioni talmente chare (sopra tutte le diffinitioni, & altri oscuri passi) che ogni mediocre ingegno, senza notitia di alcuna altra scientia sera capace de intenderla*" (my emphasis).

[3] Tartaglia 1543, Prop. I.2, 16r-v: "Il Tradottore. Molti principianti, che anchora non sanno che cosa sia il procedere scientifico demostrativo, quasi si scandalizzano di questa soprascritta propositione (per la sua bassezza) parendogli (come è il vero) puotersi essequire tal problema per la piu corta via, cioe, pigliando diligentemente con un compasso la misura della data linea *bc* & con tale appritura

commentators of Euclid who wrote in Latin, such as Clavius. Tartaglia did not either make significant changes to Euclid's text such as translated by Campanus, which remained overall abstract (with the main exception of the mention of the compass in the two first propositions). Tartaglia certainly did, in the geometrical part of the *General trattato* (Book V), offer a practical treatment of Euclidean problems, within a general teaching on the different kinds of practical geometry that exist, from the most utilitarian to the most speculative, offering thereby an example of this geometry that brings together theory and practice, as described later by Peletier. Yet, as with Peletier's *De usu*, even if this work allows to complete Tartaglia's teaching of geometry by setting forth a properly practical version of Euclid's geometrical doctrine, it was published after the edition of Euclid's *Elements*.[1] Moreover, if Euclid's problems have indeed a more practical scope than theorems, they certainly did not convey, at least in the versions of the *Elements* of Fine, Peletier or Tartaglia, the kind of practical geometry represented by Peletier's *De usu* or by Fine's *Geometria practica* or Tartaglia's *General trattato*, notably as they are always demonstrated, contrary to the way Euclid's problems were dealt with in Tartaglia or Peletier's treatises of practical geometry. As such, Euclid's problems were practical in a way that remained overall conform to the theoretical character that was conferred to his geometry and to his book of the *Elements* in the medieval and Renaissance mathematical culture.

At any rate, this does not mean that there was no underlying will on the part of such authors, or of any other sixteenth-century translator or commentator of Euclid, to bring together in some way the theoretical and the practical parts of geometry (that is, not merely the practice involved in Euclid's abstract constructions), or to better show their underlying connections. Yet, the fact of dealing with a canonical text such as Euclid's *Elements*, which was moreover acknowledged as the model *par excellence* of a demonstrative, abstract and speculative

di compasso assignarne un'altra di tal quantita che termini nel detto ponto *a* laqual cosa (per esser evidente al senso) pare a lui che non si debba, ne si possa negare. A questo se risponde, che eglie il vero che tal conclusione, per esser evidente al senso in materia, mal si puo negare: nientedimeno tal operare non seria demostrativo, & l'Autthore è tenuto à demostrar ogni sua propositione, si operativa come speculativa".

[1] While Tartaglia's translation of Euclid's *Elements* was first published in 1543, his *General trattato* was published between 1556 and 1560.

type of geometry, would have made it more difficult to realise this intention. Such an approach to geometry would certainly have been easier to implement within a practical geometry treatise, which allowed greater flexibility in terms of content, style and approach, as shown by Peletier's *De usu* or by Perez de Moya's *Tratado de geometria pratica y speculativa*.

In the development of Peletier's geometrical thought, the *De usu*, which was published fifteen years after his commentary on Euclid, could actually have been regarded as a means to realise such a conception of geometry that would bring together its theoretical and practical part. As such, even if Peletier left aside the demonstrative part and the axiomatic structure of Euclid's geometry in his treatment of Euclid's problems within the *De usu*,[1] his intention may have been to combine theoretical and practical geometry and to offer thereby a form of geometry that had a more general scope than a more classical practical geometry treatise, for which he would have chosen to entitle his *On the use of geometry* instead of "on practical geometry" (as Fine's *De geometria practica* or Bovelles's *Geometrie practique*) or "on the practice of geometry" (as Hugh of Saint-Victor's *practica geometriae* or Merliers's *Practique de geometrie*). He would, in other words, have chosen a title that could evoke the notion of practice in a more inclusive sense, encompassing the utilitarian applications of geometry as much as the processes through which geometers perform constructions and resolutions of problems in the framework of their scientific pursuit. This could be what he meant when he wrote that his intention in the treatise was to "show the practice of geometry, that is, the mechanical operations (*ouvrages mechaniques*) within theoretical geometry (*la Theorique*)".[2]

[1] The fact that Peletier did not follow the same order of propositions as in Euclid's *Elements* and left aside several of the propositions of Book I which would have been entitled to a practical treatment (such as Prop. 1, 2 and 3) is clearly justified by the fact that in the application or use of a given discipline, it is not necessary to follow the method adopted in the theory. Peletier 1572, 7: "Nunc ad Problemata veniamus: quae quidem omnia ex Euclidis Elementis desumpta sunt, non ordinatim, sed passim. *Nam in usu tradendo, non eadem methodus est necessaria quae in Theoria scribenda*"; Peletier 1573, 14-15: "Maintenant nous viendrons aux Problemes. Lesquels sont tirez des Elemens d'Euclide: non pas par ordre, mais par cy par là. *Car en monstrant l'usage de quelque Art, il n'est pas necessaire de suyvre tele methode comme si on enseignoit la Theorique*" (my emphasis).

[2] Peletier 1573, Problem 17, 32: "en ce Traité nous monstrons la pratique de Geometrie, c'est à dire les ouvrages mechaniques parmi la Theorique, comme nous avons promis dés le commencement". In the Latin version, this teaching is again only described as mechanical. Peletier 1572, 20: "Verum

Thus, for those authors who did actively introduce a more practical treatment of the *Elements*, the fact of challenging the distinction between theoretical and practical geometry in a more explicit manner in their exposition of Euclid represented a strong and significant gesture, offering a different representation of Euclidean geometry than the one more straightforwardly and more traditionally regarded as representative of theoretical geometry. This itself begs the question why they chose to follow this path, especially as they rarely offered a clear justification for their practical approach, nor did they often present an explicit discourse on the relation between theoretical and practical geometry in this context.[1]

Considering the assertion of the introductory role of Euclidean geometry with respect to practical geometry in many sixteenth-century practical geometry treatises, one possible motivation for the practical treatment of the *Elements* may have been to effectively demonstrate, from within their commentary, that Euclid's propositions (at least a certain number of them) correspond indeed to the foundation of many rules and constructions taught in practical mathematics and to show in which way. In this sense, the practical treatment of Euclid's *Elements* would in a certain manner aim to anticipate on or prepare for the teaching of practical geometry, to which theoretical geometry had often been said to introduce, helping students to better understand the principles behind the composition and use of certain instruments to measure, construct or divide magnitudes, and providing practical geometry thereby with a scientific grounding. This was in a way what Tartaglia asserted when he wrote, at the beginning of his commentary on Euclid, that Euclidean problems teach procedures useful to artisans, from painters and cartographers to sculptors and architects.[2] We can also find a similar assertion in the title of Xylander's German translation of Euclid's *Elements*, which points to the concrete uses of Euclid's geometry for artisans, such as painters, goldsmiths and carpenters, but also for those who need to know how to count and solve common arithmetical and algebraic problems by the means of the *Rechenkunst*.[3]

quia hic etiam mechanica docemus, sicut ima antè praefati sumus".

[1] Tartaglia represents in this regard an exception.

[2] Tartaglia 1543, 3v. See *supra*.

[3] Xylander 1562, title page: *Die Sechs Erste Bücher Euclidis vom anfang oder grund der Geometri. In welchen der rechte grund, nitt allain der Geometri (versteh alles kunstlichen, gwisen, und vortaili-*

Another, most probable, incentive behind the practical treatment of Euclid's *Elements*, would have been to help students gain a quicker and easier understanding of some of Euclid's concepts and propositions, notably among a public that would be less familiar with scholarly works of mathematics and theoretical mathematical concepts, notably among readers more familiar with the mechanical arts and with certain technical applications of mathematics. Indeed, the fact of teaching Euclid's propositions by instrumental means or in a more empirical manner, using concrete examples, procedures and computations, or even the fact of showing the concrete applications of some of Euclid's propositions, would enable the reader to gain a more immediate and intuitive grasp on some of the abstract constructions or universal facts concerning magnitudes, as well as a more direct identification of the reason for their truthfulness, which are otherwise mostly conveyed in a rational or logical manner, being demonstratively derived from a set of preestablished principles or previously demonstrated propositions. These practical features would also enable to show the uses and profit that could be drawn from the study of Euclid's geometry, helping the readers to apprehend the wider scope of geometrical knowledge while encouraging them to pursue their study of the *Elements* in spite of its abstractness and complexity, as was regularly claimed in the prefaces to sixteenth-century

gen gebrauchs des Zirckels, Linials oder Richtscheittes und andrer werckzeuge, so zu allerlai abmessen dienstlich) sonder auch der furnemsten stuck und vortail der Rechnenkhunst, furgeschriben und dargethon ist. (…) Alles zu lieb und gebrauch den Kunstliebenden Teutschen, so sych der Geometri und Rechnenkunst anmassen, mit vielfältiger muhe und arbeit zum trewlichsten erarnet, und in Truckh gegeben (…). It is also asserted in other passages of the prefaces, as well as in various parts of the commentary. See notably *ibid.*, b1v: "Dann wer so gar grob und unerfahren ist, daas er nit merckht wie alle kunsten (eum furnemsten aber die edlesten, so irer sinnreichen art unnd subtiler wirckhung halben den andern furgezogen werden, als da ist der Maler, Goldtschmid, Bawmaister, &c). sich mit zirckel, lineal, bleywag, ziffern und zalen begehn und behelfen muessen: dem wurde ich villeicht umb sunst und mitt vergebner muhe diese kunsten loben. Will solche bei dem pflegel unnd holtʒart bleiben lassen. Welche aber verstehn, und (wie alle kunstler, so auch nit under die gelerten, sonder under die layen gezelt werden) erfaren, wie nottwendig und nutʒ inen sei zu irem furhaben der recht gebrauch des Zirckels unnd Richtscheitts".

commentaries on Euclid's *Elements*, as those of Fine,[1] Tartaglia,[2] Forcadel,[3] Commandino,[4] Clavius.[5] This would be all the more relevant in the case of vernacular

1 Fine 1536, 2r-v: "Et proinde fit, ut nulla disciplina certior existat Geometria (...). Adde quòd usui, & commodo generis humani plurimum cedit". See also Axworthy 2016, 182-184.

2 Tartaglia 1543, 3r-3(b)r: "queste due scientie, overo discipline non hanno di bisogno di alcuna altra scientia, inquanto alla lei essentia, ma ben tutte le altre hanno bisogno di loro, come nel processo à quella lo faro cognoscer, & vedere: & non solamente le liberali, ma etiam tutte le mecanice. (...) E tanta è la utilita, oltra la soavita, dolcezza di studio che si trova nelle contemplatione mathematice, piene di certezza, che Archimede siracusano per il studio di quelle con suoi mecanici ingegni diffese un tempo la Citta di Siracusa contra l'impeto di Marco marcello Consule Romano, per ilche acquistò il nome della immortalita. Per virtu di queste Dedalo peritissimo fabricò il nominato Laberinto al Minotauro. Per mezzo di queste si fanno varii, & diversi modelli: fabricansi ponti con archi, quasi alla natura impossibile. Anchora chi con l'intelletto ben considera tutte le sorte di antique & moderne machine, & istromenti bellici, si offensivi come diffensivi, come sono bastioni, ripari, bricoli (...). Delle nove inventioni per me trovate sopra il tirar delle moderne machine tormentarie (dette dal volgo artegliarie) (...). Di quanto aiuto et presidio siano le dette due scientie, over discipline, alla Architettura,Vitruvio Pollione nel suo Prohemio lo fa manifesto. Anchora che ben considera & guarda la scientia Perspettiva, senza dubbio so trovera che nulla sarebbe, se la Geometria come matre sua non se gli accommodasse".

3 Forcadel 1564, *2r-v (speaking about mathematics in general): "Car encores qu'elles s'addressent principalement aux choses celestes, si est ce qu'elles embrassent encores les terrestres, de sorte que sans leur aide il est fort mal aisé de se tirer d'une infinité de difficultez qui embrouillent ordinairement l'esperit des hommes. Aussi par elles l'on congnoist de beau commencement tous les Royaumes, toutes les Mers, toutes les Rivieres, Montaignes, Vallées & autres choses notables, qui sont respandues sur les orizons de la terre, & ceux qui manient voz affaires sont par elles grandement secourus pour sçavoir tout incontinent de quelle importance vous peuvent estre les confederations des Princes, Roys, Republiques, & autres potentatȝ de l'Europe & de l'Asie, selon qu'ils vous ont voisins ou loingtains: lesquelles choses & autres semblables, ne se peuvent bonnement congnoistre sans la Geographie, ny la Geographie sans l'Astronomie".

4 Commandino 1572, *Prolegomena* [*4v]: "Ceterum de his hactenus summatim dixisse satis sit. Sed quoniam plerique his præsertim temporibus sola utilitate ad optimarum artium studia excitantur, liberalesque colunt disciplinas, videamus obsecro, an mathematicæ nullius sint commodi ad iuvandos humanæ vitæ usus (...). Experiantur deinde siquid dimetiri queunt absque Geodesiæ adiumento. (...) Quantum denique commodi, atque utilitatis affert Geometria, Arithmetica, & relique omnes in publicos, & privatos usus? (...) ubi vero copias ostendere cupit, ad figuram quadranguli format, nisi unius Geometriæ auxilio?"

5 Clavius 1611-1612, 7: "Dicuntur enim Geometrica elementa, eam ob causam, quod sine ipsis nullum opus Mathematicum possimus aggredi, ne dicam fructum aliquem inde percipere. Omnes siquidem Mathematicarum rerum scriptores, ut Archimedes, Apollonius, Theodosius, &c. in quis demonstrationibus usurpant haec Euclidis elementa, tanquam principia omnibus iam diu perspecta, atque demonstrata. (...) Ex his etenim elementis, veluti fonte uberrimo, omnis latitudinum, longitudinum, altitudinum, profunditatum, omnis agrorum, montium, insularum dimensio, atque divisio,

translations of Euclid's *Elements*, who could be read by people less accustomed to, and interested in, more speculative topics and demonstrations.[1]

Another possible motivation for the practical treatment of Euclid's *Elements*, which is less evident and certainly less consciously admitted in the sixteenth century than it would be in a later context, is the fact that practical geometry, insofar as it deals with the instrumental construction of curves and of resolution of problems, may be taken to represent itself the foundation of theoretical geometry with regard to the generation and construction of abstract geometrical figures. Such a conception would in a sense relate to how Newton regarded mechanics as the foundation of geometry in his *Principia mathematica*,[2] inasmuch

omnis in caelo per instrumenta siderum observatio, omnis horologiorum sciotericorum compositio, omnis machinarum vis, & ponderum ratio, omnis apparentiarum variarum, qualis cernitur in speculis, in picturis, in aquis, & in aere variè illuminato, diversitas manat.Ex his, inquam, elementis machinae totius huius mundanae est inventum medium, atque centrum; inventi cardines, circa quos perpetuo convertitur, orbis denique totius explorata figura, ac quantitas. Ostenditur atque demonstratur unius huius scientiae vi caeli universi, siderumque perennis conversio, ortus, occasus, abitus, reditus, ascensus, descensus, diei ac noctis, temporumque toto anno per omnem terrarum situm, & mundi inclinationem, varietas. Coniunctiones item planetarum, oppositiones, aspectusque varii tam expedite cognoscuntur, ut & loca illorum in caelo, & eclipses, seu Solis, ac Lunae defectiones certissime, antequam fiant, in omnes posterum tempus à Mathematicis praedici queant. Hoc denique ingens Dei, & Naturae opus, mundum, inquam, totum, mentis nostrae oculis, munere ac beneficio Geometriae subiectum conspicimus".

[1] Yet, just as there were Latin commentaries on the *Elements*, as I will show, that did introduce some practical elements within their exposition of Euclid's text, there were vernacular translations of Euclid's *Elements*, such as those of Rodrigo Zamorano in Spanish (1576), that of Federico Commandino in Italian (1575) and that of Jean Errard in French (1598) which either did not offer any additional elements of a practical nature, or in which such features were highly limited. Moreover, practical geometry treatises, which offered a more technically-minded and hands-on teaching of geometry, were published in both Latin and in the vernacular.

[2] Newton 1687, i; Newton (transl. in Cohen and Whitman 1999, 381-382), *Preface to the reader*: "For the description of straight lines and circles, which is the foundation of *geometry*, appertains to *mechanics*. *Geometry* does not teach how to describe these straight lines and circles, but postulates such a description. For *geometry* postulates that a beginner has learned to describe lines and circles exactly before he approaches the threshold of *geometry*, and then it teaches how problems are solved by these operations. To describe straight lines and and to describe circles are problems, but not problems in *geometry*. *Geometry* postulates the solution of these problems from *mechanics* and teaches the use of the problems thus solved. And *geometry* can boast that with so few principles obtained from other fields, it can do so much. Therefore *geometry* is founded on mechanical practice and is nothing other than that part of *universal mechanics* which reduces the art of measuring to exact propositions and demonstrations. But since the manual arts are applied especially to making

as it teaches the modes according to which geometrical figures come about from the motion of a point, a line or a surface; or, closer to the sixteenth century, it would relate to the foundational role given by Descartes to curve-tracing instruments in his new geometry.[1] This could, for instance, be implicitly set forth in the commentary on the *Elements* of Christoph Clavius, who explicitly related Euclid's definition of the straight line, and indirectly the first two postulates which govern the construction of all straight lines in Euclid's geometrical work, to the use of the straightedge.[2] In this regard, just as theoretical geometry would have been regarded as the foundation for practical geometry by demonstrating the truth of the taught procedures, practical geometry would have constituted the foundation of theoretical geometry by providing the instrumental mode of generation of geometrical objects, these having been only afterwards dealt with in an abstract manner and without any reference to instruments. If not an incentive, the admission of the foundational function of practical geometry with regard to the constitution of geometrical objects could have been a consequence of this more explicitly instrumental interpretation of Euclidean geometrical concepts. Along with the fact of allowing a numerical treatment of magnitudes, the fact of offering a more practical treatment of theoretical geometrical propositions may have therefore partially contributed to the changes in the conception of geometrical knowledge that would lead to Descartes's new approach to geometry.

In a quite different, but not unrelated way, the fact that geometry in general was held to have come from the practice of land-measurers, even if it was said to have progressed beyond it, could (as said) have somewhat enabled to justify the fact of dealing with Euclid's text in a more practical manner, by implicitly setting forth the origin, as well as the use, of certain geometrical principles or propositions in concrete problems or practices. This would differ from the fact of appealing to concrete examples in purely pedagogical aim, that is, when

bodies move, *geometry* is commonly used in reference to magnitude, and *mechanics* in reference to motion". (Emphasis proper to the translation). See Arthur 2021, 297, and Guicciardini 2009, 293-299.

[1] On the connection between the generation of curves and instrumental processes in Descartes's geometry, see notably Molland 1976 and Bos 1981.

[2] Axworthy 2022, 199-202 and 213-217. The same could be said of the use of the compass in relation to the definition of the circle and to Euclid's third postulate, but this was not made as explicit in Clavius' commentary on the *Elements*.

aiming to facilitate the learning of practical geometry, since it would rather function as a historical explanation for the nature and structure of certain geometrical problem. Yet, since Euclidean geometry is considered to have left aside any immediate utilitarian concerns to focus on the properties and relations of figures in a universal and demonstrative manner, it does not seem that it could have been attributed a significant role (as opposed to the origin of the constitution of geometrical figures in instrumental processes) in the foundation of geometry as an abstract science of magnitudes and thus in the comprehension of the concepts, order and constitution of Euclidean geometry, standing rather as an exterior justification for its historical existence and its ulterior uses.

An equally possible motivation, which does not necessarily conclude the list of possible incentives for the practical treatment of Euclidean propositions, could have been the desire, which was expressed by Peletier in the preface of *De l'usage de geometrie*, to conjoin the practical and theoretical parts of geometry so as to obtain a unified and comprehensive knowledge of magnitudes in all its aspects, in which knowledge and experience, or contemplation and application, are appealed to in order to bring each type of geometrical knowledge, theoretical and practical, to perfection. As said, this is not so much the image that may be derived from Peletier's treatment of Euclid's *Elements*, but it may (as said) have been part of a later project on his part. In a certain manner, one may find a certain will to unify theoretical and practical geometry in other works of practical geometry which offered a preliminary teaching on the theoretical principles of geometry, such as Fine's 1532 *Geometria libri duo*, Bovelles's *Gometrie practique*, Tartaglia's *General trattato*, Perez de Moya's *Geometria practica y speculativa*, and even Errard's *Geometrie et practique generalle d'icelle*, even if then the content of the theoretical part mainly consists in a long list of definitions.

In a more conspicuous manner, such an intention may be regarded as materialised by Petrus Ramus' 1569 *Geometria*,[1] in which elements of theoretical and practical geometry are brought together to the point of making both parts difficult to distinguish and in which the place given to demonstrations is sensibly restricted, allowing for a more empirical, instrumental and numerical modes of apprehension of geometrical notions. Yet, Ramus' aim was very different from those of the authors presented above in the sense that, to him, geometry as

[1] Ramus 1569. On this work and its history, see Goulding 2018.

a whole should be conceived as an art of measuring, calling it an *ars bene metiendi*,[1] which as such would have a chiefly practical aim and would be in priority addressed to craftsmen.[2] Indeed, as shown in particular by his *Scholae mathematicae*, Ramus actually aimed to overthrow the Euclidean model of mathematical teaching, which, he claimed, had been contaminated by the Platonic representation of mathematics as detached from the material world. In this context, the aim and scope of geometry is said to be best represented by its use in the art and practice of those who apply its precepts, namely astronomers, geographers, surveyors, navigators, engineers, architects, painters and sculptors, than by the study of its principles.[3]

Looking at the commentators on Euclid that did propose a more obviously practical treatment of Euclid's *Elements*, and with which I will deal in the following pages, it is unlikely that they intended to offer thereby a renewed form of geometry in which theoretical and practical geometry would be conjoined in the sense that the way in which these two parts of geometry were connected was still very unbalanced, the practical elements representing mostly a limited part of these works, being often confined to a specific part of the commentary. Yet, such a conception may have been derived *a posteriori* from this practical treatment of Euclid's *Elements* and its greater frequency towards the end of the sixteenth century, since it presents to us a certain shift in the treatment of the distinction between theoretical and practical geometry in the direction of a certain reduction of its importance and significance in the representation

[1] Ramus 1569, *Geometria libri XXVII*, 2: "*geometria hunc in modum definita sit ars bene metiendi.* Nomen autem re ominata levius est. Geometria enim dicitur tanquam terrae tantum dimetiendae ars quaedam sit, quod nomen videtur in Aegypto primum factum esse: ubi ad terminos agrorum Nili inundationibus obrutos restituendum primum geometria adhibita sit" (my emphasis).

[2] Loget 2019 and Goulding 2006.

[3] Ramus 1569, *Geometria libri XXVII*, 1: "*Geometria est ars bene metiendi.* Finis geometriae est bene metiri, ideoque suo fine definitur: Bene metiri igitur est cuiusque rei mensurabilis naturam atque affectionem considerare, resque mensurabiles comparare inter se, rationemque & proportionem atque similitudinem perspicere: id enim totum est bene metiri, sive congruentia & applicatione datae mensurae, sive multiplicatione terminorum, sive facti per multiplicationem partitione, sive quacunque alia ratione rei mensurabilis affectio consideretur. Atque hic finis geometriae usu atque opere geometrico multò splendidior apparebit, quam praeceptis, cum animadvertes astronomos, geographos, geodetas, nautas, mechanicos, architectos, pictores, statuarios in descriptione & dimensione astrorum, regionum, fundorum, machinarum, aequaorum, aedificiorum, tabularum, signorum nihil aliud quàm geometria uti".

and teaching of geometry, and of mathematics in general, altogether in terms of content, style and addressed audience. This, in a way, points to the evolution of practical geometry observed by D. Raynaud, according to which the category and designation of practical geometry will start to disappear from the end of the nineteenth century, its content and approach to geometry being absorbed by manuals of elementary geometry and by more specific types of professional mathematical knowledge.[1]

It nevertheless remains that, in the sixteenth century, the introduction of practical features in commentaries on Euclid's *Elements* were, as said, much more present than in other works belonging to the ancient and medieval Euclidean tradition and were intended as practical in a much more evident and explicit manner. And given the correspondence commonly admitted at the time between Euclid's *Elements* and theoretical geometry, and more generally between Euclid's *Elements* and geometry in general, this would constitute a crucial step in the transformation of the concept of geometrical knowledge towards an erosion of the distinction between practical and theoretical geometry, at least within the more classical framework according to which Euclidean geometry was held as representative of theoretical geometry.

Hence, beyond the question of the motivations behind the practical adaption of Euclidean propositions, it is important to look in more detail at the different ways and the extent to which they actually reworked Euclid's text or explained it in a more practical manner, and which representation it provides of the connection between theoretical and practical geometry in this context. More generally, it is important to see in which way they might have contributed to change the relation and status of practical and theoretical geometry, and the representation of geometry in general thereby.

In the next part of this article, I will therefore show through more specific examples some of the different ways in which certain characteristics proper to medieval and Renaissance Western practical geometry treatises could be found within translations and commentaries on the *Elements* in the sixteenth century. I will attempt to manifest more concretely how it may have contributed to operate a form of hybridization internal to the domain of geometry, impacting not only the form and content of Euclid's treatise, but also the place attributed

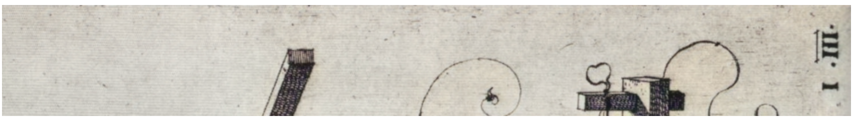
[1] Raynaud 2015, 119-124.

to Euclidean geometry within the spectrum going from applied geometry to speculative or theoretical geometry, and thereby the very representation of the scope, nature and methods of geometry at the end of the sixteenth century.

2. The practical treatment of Euclid's Elements in the sixteenth century

My intention in the following pages is to present the main ways in which Euclidean principles and propositions were treated according to a practical approach in sixteenth-century translations and commentaries of the *Elements*, namely: 1) through a practical adaptation of Euclid's text itself, such as transmitted by the version of Campanus of Novara published in 1482 and by the 1505 Latin translation from the Greek of Bartolomeo Zamberti, 2) through the addition of commentary sections specifically dedicated to the practical interpretation of a given proposition, 3) through an empirical handling of Euclidean demonstrative methods, 4) through a numerical treatment of Euclidean propositions and 5) through explicit references to artisanal applications.

I will not be able to provide here a complete picture of these different practical approaches or features by quoting the various ways in which each of them appeared throughout the sixteenth-century Euclidean tradition,[1] but simply provide one or two examples taken from the most conspicuous cases. I will provide along the way examples drawn from contemporary practical geometry treatises as a means to better manifest the relation between the considered examples and this practical geometry tradition.



[1] This will be the object of a forthcoming study.

Table 1: Scheubel 1550, Prop. I.3, construction 1.

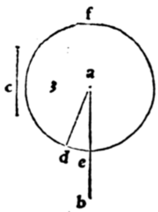
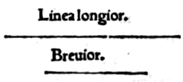
	Euclid [a]	*Scheubel*
ENUNCIATION [b]	Given two unequal straight lines, to cut off from the greater a straight line equal to the less.	Given two unequal straight lines, to cut off from the longer a straight line equal to the shorter. There are three operations, or constructions, for this proposition.
EXPOSITION	Let AB, C be the two given unequal straight lines, and let AB be the greater of them.	
SPECIFICATION	Thus it is required to cut off from AB the greater a straight line equal to C the less.	
CONSTRUCTION	At the point A let AD be placed equal to the straight line C; [i. 2] and with centre A and distance AD let the circle DEF be described. [Post. 3] 	First, let the quantity of the smaller line be taken by the means of a compass. Then, let it be marked as a point in the longer, starting from one of its extremities, and the task will have been done, 
DEMONSTRATION	Now, since the point A is the centre of the circle DEF, AE is equal to AD. [Def. 15] But C is also equal to AD. Therefore each of the straight lines AE, C is equal to AD; so that AE is also equal to C. [C.N. 1]	which can be demonstrated from the common notion: things that are equal to one thing are equal to each other. [c]

Table 1: Scheubel 1550, Prop. I.3, construction 1.

CONCLUSION	Therefore, given the two straight lines AB, C, from AB the greater AE has been cut off equal to C the less. (Being) what it was required to do. [d]

[a] Euclid, Prop. I.3 (Heath 1956, I, 246-247). The diagram is taken from Zamberti, in Lefèvre 1516, I.3, 6v. [b] The parts of the propositions are here distinguished and designated according to the nomenclature used by Proclus, in his commentary on the first book of Euclid's *Elements* (Proclus, Friedlein 1873, 203; transl. in Morrow 1992, 159): enunciation | πρότασις, exposition | ἔκθεσις, specification / διορισμός, construction | κατασκευή, proof | ἀπόδειξις and conclusion / συμπέρασμα. [c] Scheubel 1550, I.3, 84: "ΠΡΟΤΑΣΙΣ Δύο δοθεισῶν εὐθειῶν ἀνίσων, ἀπὸ τῆς μείζονος, τῇ ἐλάσσονι ἴσην εὐθεῖαν ἀφελεῖν. — Duabus datis rectis lineis inaequalibus, à longiori, breviori aequalem rectam lineam abscindere. Est huius propositionis triplex operatio, seu fabrica. Prima, ut officio circini quantitas brevioris accipiatur: ea deinde in longiore, ab extremitate una incipiendo, puncto aliquo signetur: & factum erit negotium, id quod per communem illam noticiam, Quae uni sunt æqualia, et inter se sunt æqualia, demonstrari poterit". Cf. Zamberti 1505, A3r: "Duabus datis rectis lineis inaequalibus: a maiori minori aequalem rectam lineam abscindere. Sint datæ duæ rectæ lineae inaequales AB, C quarum maior sit AB. Oportet ab ipsa AB maiori ipsi C minori æqualem rectam lineam abscindere. Ponatur per secundam propositionem ad signum A lineæ vero et quam A signum centrum est circuli DEF equalis est AE ipsi AD. At linea C ipsi AD est æqualís. utraque igitur & AE & C ipsi ad est æqualis. Quare & linea AE ipsi C est aequalis. Duabus igitur datis rectis lineis inaequalibus AB, C ab ipsa AB maiori ipsi C minori aequalis abscisa est AE quod facere oportebat".
[d] Euclid, Prop. I.3 (Heath 1956, I, 246-247).

2.1. Practical adaptation of Euclid's proofs

In the 1550 Latin commentary by the Tübingen university professor Johannes Scheubel, the classical proof of Euclid's Prop. I.3[1] is for instance replaced by

[1] Euclid, Prop. I.3 (Heath 1956, I, 246): "Given two unequal straight lines, to cut off from the greater a straight line equal to the less".

a) *b*)

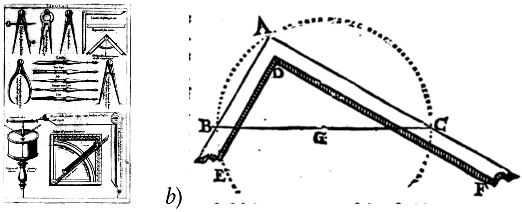


Figure 1: *a*) Pomodoro, *Geometria practica*, 1599, Tavola I; *b*) Clavius, *Euclidis elementorum libri XV: Accessit XVI*, 1611-1612, Df. I.12, p. 17.

three different modes of construction, or *operationes* or *fabricae*. Already, the fact of providing different modes of construction is practical in the sense that it does not follow one determinate argumentative model (as in the more classical versions of the *Elements*) and offers the possibility to choose the construction that is most appropriate to a given situation. It thus places the circumstances in which the construction is to be performed above the demonstration of its geometrical validity and its function in Euclid's axiomatic system.

First construction

Among the three proposed constructions, the first (Table 1) is the most practical.

Instead of requesting, as in Euclid's classical proof, that a line equal to the shorter given line be placed at one extremity of the longer (through Euclid's Prop. I.2) and then to draw a circle according to its length that will cut off, from the longer line, a segment equal to it (through the third postulate), Scheubel invites the reader to measure the length of the shorter line with a compass and to mark a point in the longer line according to this interval with the mobile leg of the compass.

The fact of referring to instrumental procedures within the proofs of Euclid's propositions, even if present in certain medieval versions of the *Elements* (as in Campanus), contrasts (as said) with the total absence of references to instruments in the Greek text of Euclid and its Greek-based Latin translations. The Greek text of Euclid's *Elements*, which was made available in print by Simon

Grynaeus in 1533, was very likely known to Scheubel since he systematically added the Greek text of the enunciations to the Latin propositions. On the other hand, as said, the use of instruments is a feature that is indissociable from practical geometry, even if the instruments of practical geometry went far beyond the compass and straightedge useful to Euclidean constructions, as they mainly included measuring and surveying instruments.

Now, the number of references to instruments in commentaries on Euclid's *Elements* increased in the second half of the sixteenth century (see Figs. 1, 2), and were appealed to in a more systematic manner, as in the commentaries of Henry Billingsley and Christoph Clavius.[1] In the concerned texts, references to instrumental procedures were sometimes supplemented with illustrations of geometrical instruments, instructions to build some of the mentioned instruments, specifications regarding their material features, the conditions of their use in a concrete context and references to artisanal versions of the geometer's instruments (Fig. 1).[2]

[1] See *infra*. In Xylander 1562, see for instance the title page: *Die Sechs Erst Bucher Euclidis, vom anfang oder grund der Geometri. In welchen der rechte grund, nitt allain der Geometri (versteh alles kunstlichen, gwisen, und vortailigen* gebrauchs des zirckels, Linials oder Richtscheittes und andrer werckzeuge*, so zu allerlaj abmessen dienstlich) sonder auch der furnemsten stuck und vortail der Rechenkhunst, furgeschriben und dargethon ist* (my emphasis).

[2] In Tartaglia 1543, Post. 1, 12r: "Et che'l sia il vero, el si sa che communemente per tirar, over designare le dette linee di puoca longhezza, el *si costuma prima di farsi fare una listetta di legno, overo di alcuno metallo piu plana & retta che sia possibile*, & secondo l'ordine di quella tirale dette linee rette da un ponto ad un'altro, secondo le sue occorrentie, laquale listetta alcuni la chimano Rega, & alcuni altri Regola, laqual rega, over regola, essendo perfettamente giusta pur piu giustamente tirara le dette linee rette, *domente che la superficie della materia dove se tirano sia perfettamente piana, & che lui sia anchora diligentissimo nell' operare*: lequal cose *non è molto facile accordarle*, cioe, che la regola sia perfettamente piana, & retta, & *che la superficie della materia dove che si tirano sia similiter perfettamente piana, & che l'operante usi tutta quella perfetta diligentia che si possa usare.* Similmente per tirare, over designar le linee di molta longhezza *costuma di tuorre una corda sottile longa à sofficientia, & imbratta quella con una spongia infusa in certa acqua tinta communemente d'un colore rosso*, & lui insieme con un compagno tirano la detta corda, & ciascaduno di loro con una mano la firmano l'uno all' altro, dapoi l'uno di loro con l'altra mano *tira, & marca sforzatamente la detta corda rettamente in aere*, dapoi la lascia scorrerere, & *quella percuottendo nella superficie di quella materia, dove si ritrova, vi lascia la linea signata di quel suo liquore*"; Dee, in Billingsley 1570, XII.17, 380r: "(...) shall you in this delineation in apt pastborde, or like matter framed, finde al things in this probleme very evident. I neede not warne you, that the line AY may easely be imagined, or with a fine thred supplyed (...)"; Clavius, *Euclidis elementa...*, 1611-1612, Df. I.12, 17: "Facilius idem cognoscemus beneficio *normae alicuius accurate fabricatae*, qualem refert instrumentum ABC,

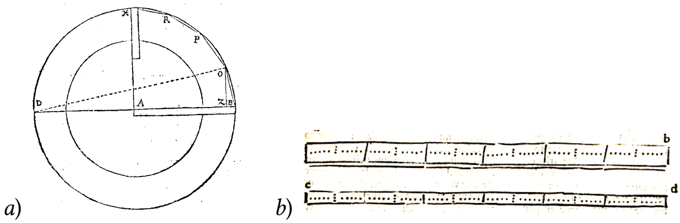


Figure 2: *a*) Billingsley, *The Elements of geometry*, 1570, XII.17, 380r; *b*) Tartaglia, *Euclide megarense philosopho*, Df. I.2, 1543, 7r.

Also notable in Scheubel's treatment of Prop. I.3 is the replacement of Euclid's demonstration by the sole mention of Common Notion 1,[1] through which Scheubel merely hinted at the way the demonstration should be conducted ("this can be demonstrated from the common notion: things that are equal to one thing are equal to each other"). This approach is comparable to the use of "directions for proof" (i.e. mere indications of the principles or propositions that allow to prove the propositions) in the medieval compilations of the *Elements* such as that by Robert of Chester (otherwise called Adelard II).[2]

The reduction of the demonstration here, which will end in a total suppression in the third construction, also recalls the treatment of Euclidean proposition in practical geometry treatises, in which the proofs were most often stripped of the demonstrative part, as shown here through the treatment of Euclid's Proposition I.11[3] in Robert Recorde's *Pathway to Knowledge*[4] (Recorde 1551, *The*

constans duabus regulis AE, AF, ad angulum rectum in A, coniunctis".

[1] Euclid, CN 1 (Heath 1956, 155): "Things which are equal to the same thing are also equal to one another".

[2] Murdoch (1968) and Busard (2005, 4 and 6). See, for example, Prop. I.2 (Busard 1992, 109): "Deinde ex circuli descripcione atque ex tercia et prima comuni concepcione argumentum elice" and Prop. I.3 (*ibid).*: "Deinde ergo ex circuli descripcione argumentum elicito".

[3] Heath 1956, I, 269. The diagram is taken from Zamberti, in Lefèvre 1516, I.11, 9r.

[4] Although this work mainly only contains an adaptation of the first four books of Euclid's *Elements*, its practical scope is made clear in the title. Recorde 1551, title page: *Pathway to knowledg, containing the first principles of Geometrie, as they may moste aptly be applied unto practise, bothe for use of instrumentes Geometricall, and astronomicall and also for projection of plattes in everye kinde, and therfore much necessary for all sortes of men.* Moreover, it initially intended to go beyond

fifth conclusion, c2v),[1] Digges' *Pantometria* (Digges 1571, *The first Chapter*, b4r-v.), Peletier's *De l'usage de geometrie* (Peletier 1573, II, 16),[2] and Perez de Moya's *Geometria practica y speculativa* (Perez de Moya 1573, 23-24).[3]

the adaptation of Euclid's books on plane geometry, since in the table of content, in which four books are announced (*The argumentes of the foure bookes*), the third and fourth book should have taught various constructions and applications of Euclid's problems, which he called conclusions, as well as procedures to measure areas and volumes of any surface and body. (*The third booke intreateth of divers formes, and sondry protractions thereto belonging, with the use of certain conclusions. The fourth booke teacheth the right order of measuringe all platte formes, and bodies also, by reson Geometricall*).

[1] The text quoted in the table corresponds to the specific example of a more general explanation, which presents the construction without referring it to a specific figure. The text of this part can be found below, 55 fn. It is then followed by an alternative case, when the point is situated close to one of the extremities of the given line segment (c2v-c3r): "Howe bee it, it happeneth so sommetymes, that the pricke on whiche you would make the perpendicular or plum line, is so nere the eand of your line, that you can not extende any notable length from it to thone end of the line".

[2] "D'un point donné en une ligne droitte, tirer une ligne perpendiculaire. Soit la ligne droitte AB, & le point en elle donné, soit C: duquel point il faille mener une ligne perpendiculaire. Je fay que le point C donné, soit le milieu de la ligne, ce qui se fera en descrivant un cercle sur iceluy point C, de l'estendue de la plus grande portion, sçavoir est de CA, & allongeant CB, jusqu'à la circonference, si que CD, soit egale à la portion AC. Adonque sur l'extremité A, je mets le piè ferme du compas, & descri un Cercle: lequel soit de plus grande estendue, que n'est la demie AC. Puis le compas demeurant en son ouverture, je descri sur l'autre extremité D, un Cercle egal au premier, et qui l'entrecoupe au point E. Finablement du pt E, je tire une ligne au point C donné: qui sera la ligne EC, perpendiculaire à la ligne AB donnée: c'est assavoir, que chacun des deux angles ACE & BCE sera droit. Ce qu'avions proposé faire". Cf. Peletier 1572, Problem II, 8: "Datam Lineam rectam bifariam secare. Sit recta linea AB, quae bifariam, hoc est in duo aequalia, secanda sit. Super duobus extremis A & B, describo duos Circulos aequali intervallo, maiore tamen quàm sit dimidia pars ipsius AB datae (nam siquis in hoc haereat, possunt Circuli duci secundum intervallum totius AB) Atque ij omnino se intersecabunt in duobus punctis oppositis, ut in C & D. Tum ab una intersectione ad alteram, duco lineam rectam CD & erit ea quae secabit lineam AB datam bifariam, in puncto E, Quod facere oportuit".

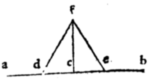
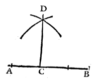
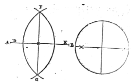
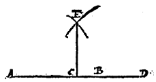
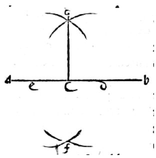
[3] "Capi. X. muestra de un punto propuesto en una linea, sacar otra que cayga perpendicular, o derecha, ò en angulos rectos, sobre el punto dado, en la dicha linea. Sea la linea dada AB y el punto, ò señal do ha de caer la otra linea perpendicular sea el punto C, abre el compas en la distancia que quisieres, y assienta el un pie en el punto C y con el otro a una parte y otra del dicho punto C haz dos señales, como los dos puntos DE. Luego abre el compas mas, en la quantidad que te paresciere, y assienta el un pie en el punto E y con el otro en la parte alta, y baxa de la dicha linea señala un pedaço de circunferencia, y luego buelve à poner el pie del compas en el otro (punto D y estandose en la misma abertura) haz en la parte alta y baxa otra poca de circunferencia de modo que se corte con las otras que heziste, como mustran el punto F y el punto G. Luego echa una raya desde el punto G al punto F y passar ajustamente por el punto C de la linea AB (que fue el lugar señalado) y por

Table 2: Prop. I.11 of Euclid's *Elements*

Euclid	*Recorde 1551*	*Digges 1571*	*Peletier 1573*	*Perez de Moya 1573*
To draw a straight line at right angles to a given straight line from a given point on it.	To make a plumme line or any pricke that you will in any right lyne appointed.	Howe Perpendiculares uppon any straight line are erected.	To draw a perpendicular line from a given point in a given straight line.	(...) how to draw, from a proposed point in a line, another line which falls perpendicularly, or vertically, or at right angles, on the given point in that line.
Let AB be the given straight line, and C the given point on it. [EXPOSITION]	The lyne is AB the prick on whiche I shoulde make the plumme lyne, is C.	Admit AB were the line to be crossed,	Let there be the straight line AB and let the given point on it be C,	Let the given line be AB and let the point, or sign,
Thus it is required to draw from the point C a straight line at right angles to the straight line AB. [SPECIFICATION]		that ye desired a Perpendiculare or plumbe line in C,	wherefrom the perpendicular line is to be drawn.	from which the other line should be made to fall perpendicularly, be the point C.
Let a point D be taken at random on AC; let CE be made equal to CD; on DE let the equilateral triangle FDE be constructed, and let FC be joined. [CONSTRUCTION]	then open I the compasse as wyde as AC, and sette one foote in C, and with the other doo I marke out CA and CB, then open I the compasse as wide as AB, and make ij. arch lines which do crosse in D, and so have I doone.	open your compasse, put the one foote in C, make of either side the line one pricke, DE. Nowe extend the compasse to the widenesse of bothe, or shorter, putting the one foote in D and the other immoveable, making an arcke over and under C.	I make the given point C the middle of the line, which will be done by describing a circle around this point C, according to the length of the greatest portion, that is, of CA, and by extending CB to the circumference, so that CD be equal to the portion AC. Hence, on the extremity A, I place the immobile foot of the compass and describe a circle, which will be of greater extent than the half[-segment] AC.	Open the compass according to any distance and place one of the feet on the point C, and with the other make two marks on one and the other parts of the point C, such as the two points D, E. Then open the compass according to a greater extent, of the quantity which appears good to you, and place one of the feet on the point E, and with the other, draw out a piece of circumference in the parts higher and lower of the line,
		This done discretely, remove the compasse from that Centre to E (remaining so opened) there fixe one foote, with the other crosse the arke afore made above and beneath C, where make two points, or these letters FG.	Then, the compass remaining with its opening, I describe on the other extremity D, a circle equal to the first, and which will intersect it at the point E.	and then place again the foot of the compass on the other point (the point D, and remaining with the same opening), make in the higher and lower parts another small piece of circumference so that it intersects with the others that were made, as show the point F and the point G.
		Then take a Ruler and lay him upon bothe the poyntes crossing the centre C. Thus drawe your plumbe or squire line FCG.	Lastly, from point E, I draw a line to the given point C, which will be the line EC, which is perpendicular to the given line AB, that is to say, that each of the two angles ACE and BCE will be a right angle.	Then draw a line from the point G to the point F and pass precisely through the point C of the line AB (which is in the marked out place)

consiguiente la tal linea sera perpendicular, y hara dos angulos yguales con la dada linea AB como se demuestra por la II del I lib. de Eucli".

Table 2: Prop. I.11 of Euclid's *Elements*

	 In like manner any line may be devided in halfe, or circle in four equall parte. See the example on the side folowing.			
Euclid: I say that the straight line FC has been drawn at right angles to the given straight line AB from C the given point on it. For, since DC is equal to CE, CF is common, the two sides DC, CF are equal to the two sides EC, CF respectively; and the base DF is equal to the base FE; therefore the angle DCF is equal to the angle ECF; and they are adjacent angles. But, when a straight line set up on a straight line makes the adjacent angles equal to one another, each of the equal angles is right; therefore each of the angles DCF, FCE is right. [DEMONSTRATION]				
Therefore the straight line CF has been drawn at right angles to the given straight line AB from the given point on it. Q. E. F. [CONCLUSION]			Thus each of the two angles, ACE and BCE, will be right angles. What was to be done.	and therefore this line will be perpendicular and will have two equal angles with the given line AB, as was demonstrated by the eleventh proposition of the first book of Euclid.

Indeed, when dealing for example with Prop. I.11, none of these practical geometry treatises included Euclid's demonstration or referred to the principles and the propositions on which this construction depends, contrary to how it was done in classical editions of Euclid's *Elements.* The construction itself is expanded and constitutes the bulk of the proposition. It is also relatively different from one text to the other, and a fortiori from Euclid's original construction. One common feature however is the abandonment of the equilateral triangle, which Euclid needs to prove that the construction is geometrically valid. Since the proof has been left aside, it is no longer needed. Indeed, in most of these cases, the intersection of compass arcs that are used to produce the line would not necessarily enable to construct an equilateral triangle, but an isosceles triangle, which both simplifies and generalises the actual execution of the construction.[1] And in all these texts, the steps of the instrumental procedure, through which the construction may be concretely realised, are made explicit.

[1] In Recorde's text, the construction of the equilateral triangle certainly remains in the preliminary general exposition of the construction (though without a diagram), but it is removed when applied to the specific case (presented in the comparative table). See Recorde 1551, The fifth conclusion (= the fifth problem, or Prop. I.11), c2v: "To make a plumme line or any pricke that you will in any right lyne appointed. Open youre compas so that it be not wyder then from the pricke appoynted in the

Second construction

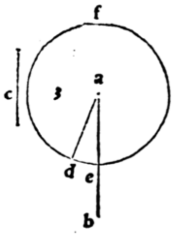
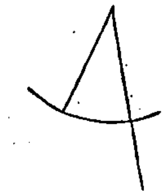
For his second construction (Table 3), Scheubel proposes instructions that are closer to those of Euclid, insofar as a circle is drawn after the extremities of the two lines have been conjoined and in the sense that he mentions one of the postulates on which this construction depends (i.e. the third postulate). However, it is not a line *equal* to the shorter line that is joined to the longer line, but the actual given shorter line ("when the two proposed lines will somehow have been conjoined by their extremities"), which implies that it has been somehow moved toward the longer line through a mechanical rather than through a geometrical method, that is, not according to a mode of operation authorised by Euclid's prior propositions or constructive postulates.[1] This attitude, as in the first construction, itself marks a practical approach, since Scheubel tends to free himself from the necessity of grounding his construction on Euclid's principles and thus to demonstrate its geometrical validity.

In line with this attitude, Scheubel here specifies that only an arc of circle may be drawn ("let a circle, or only an arc in place of the circle, be described"), as is illustrated by the diagram. This diagram, in addition to leaving aside the full circles required by Euclid's proof, displays process traces (or compass arcs), in the style of the diagrams often featured in practical geometry treatises (Fig. 3).

line to the shortest ende of the line, but rather shorter. Then sette the one foote of the compasse in the firste pricke appointed, and with the other fote marke ij other prickes, one of eche syde of that fyrste, afterwarde open your compasse to the wydenes of those ij new prickes, and draw from them ij arch lynes, as you did in the fyrst conclusion, for making of a threlyke triangle. Then if you do mark their crossing, and from it drawe a line to your fyrste pricke, it shall bee a iust plum lyne on that place".

[1] Euclid, Post. 1-3 (Heath 1956, 154): "To draw a straight line from any point to any point; To produce a finite straight line continuously in a straight line; To describe a circle with any centre and distance". Previously demonstrated propositions: Prop. I.1 (Heath 1956, 241): "On a given finite straight line to construct an equilateral triangle and Prop. I.2 (Heath 1956, 244): "To place at a given point (as an extremity) a straight line *equal* to a given straight line" (my emphasis).

Table 3: Scheubel 1550, Prop. I.3, construction 2.

Euclid	*Scheubel*
At the point A let AD be placed equal to the straight line C; [i. 2] and with centre A and distance AD let the circle DEF be described. [Post. 3] 	The second is when the two proposed lines will somehow have been conjoined by their extremities, then, by the third postulate, let a circle, or only an arc in place of the circle, be described from their point of conjunction, that is, which cuts the longer straight line according to the quantity, or according to the interval, of the shorter, and the same will have been accomplished. 
Now, since the point A is the centre of the circle DEF, AE is equal to AD. [Def. 15] But C is also equal to AD. Therefore each of the straight lines AE, C is equal to AD; so that AE is also equal to C. [C.N. 1]	The demonstration [of this operation] is the above-given definition of the circle, since the lines that fall from the centre to its circumference are equal to each other, by the same [definition].[b]

[a] Euclid, Prop. I.3 (Heath 1956, I, 246-247): "Secunda est, ut lineae propositae duabus suis extremitatibus utcunque coniungentur, secundum quantitatem deinde, vel intervallum brevioris, ex coniunctionibus puncto, per tertium postulatum, circulus, vel arcus, vel arcus tantum circuli loco, qui tamen longiorem rectam secat, describatur: & idem effectum erit. Huius autem demonstratio est ipsa circuli definitio supra tradita, cum lineae à centro in circumferentiam cadentes, per eandem, inter se sint aequales".

[b] Scheubel 1550, I.3, 84: "Secunda est, ut lineæ propositæ duabus suis extremitatibus utcunque coniungentur, secundum quantitatem deinde, vel intervallum brevioris, ex coniunctionis puncto, per tertium postulatum, circulus, vel arcus tantum circuli loco, qui tamen longiorem rectam secat, describatur: & idem effectum erit. Huius autem demonstratio est ipsa circuli definitio supra tradita, cum lineæ à centro in circumferentiam cadentes, per eandem, inter se sint æquales".

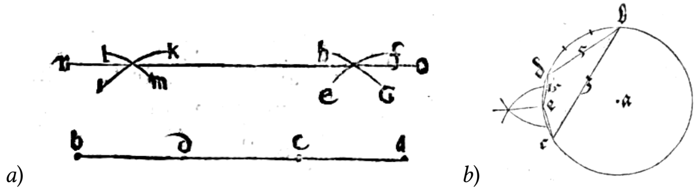


Figure 3: *a*) Perez de Moya, *Tratado de geometria practica*, 1573; *b*) Dürer, *Underweysung der Messung*, 1525.

As was shown by E. Lee,[1] such diagrams, that displayed process traces in the form of compass arcs, became increasingly present within sixteenth-century printed editions of Euclid's *Elements*. Lee interpreted this (accurately, I believe) as a witness of the influence of practical geometry on Euclidean geometry in the early modern era.[2]

Similarly to the first construction, Scheubel's proof is then reduced to a mere reference to Euclid's definition of the circle ("The demonstration is [founded on] the above-given definition of the circle").

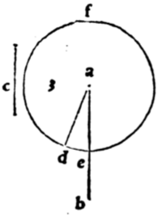
Third construction

If Scheubel's third construction (Table 4) is more faithful to Euclid's construction, insofar as it appeals to Prop. I.2 to place, at one extremity of the longer line, a line that is *equal* to the given shorter line, he then only refers to the construction taught in the second operation and does not provide any demonstration.

In addition to the fact that Scheubel's treatment of Prop. I.3 offers the possibility to choose the construction that is most appropriate to one's situation, what evokes here the treatment of Euclid's propositions found in practical geometry treatises is not only the direct appeal to instrumental procedures and the reduction or suppression of the proof, but also the freedom taken with the structure

[1] Lee 2018. See also Lee 2020, 488-498.
[2] Lee 2018, 12-14 and Lee, 533-536.

Table 4: Scheubel 1550, Prop. I.3, construction 3.

Euclid[a]	*Scheubel*
At the point A let AD be placed equal to the straight line C; [i. 2] and with centre A and distance AD let the circle DEF be described. [Post. 3] 	The third operation [requests] that, by the preceding second proposition, a line equal to the shorter line first be led from any of the extremities of the longer, just as from any given point, and then, according to what the second operation requires, let a line equal to it be cut off from the longer, and, thirdly, what the proposition requested has been done.[b] 

[a] Euclid, Prop. I.3 (Heath 1956, I, 246-247). [b] Scheubel 1550, I.3, 84: "Tertia huius operatio est, ut, per præcedentem propositionem secundam, primò ab extremitate longioris alterutra, tanquam à puncto aliquo dato, linea breviori aequalis educatur: atque huic deinde à longiore, prout secunda huius operatio exigit, æqualis abscindatur, & tertiò, quod volebat propositio, factum erit".

of the proposition. This is also shown by the fact that Scheubel systematically suppressed the lettering of the diagram.[1]

2.2. Addition of commentary sections specifically dedicated to the practical treatment of a given proposition

Among the authors who added to a commentary section specifically dedicated to the practical treatment of the proposition, after providing Euclid's classical proof, was Niccolò Tartaglia, in his 1543 Italian translation and commen-

[1] Scheubel chose not to use any lettering for his diagrams, which is something he announced already in the title of his work (*absque literarum notis*). In his prefatory epistle (Scheubel 1550, a3r-4r), he justified this by claiming that it is more conform to the approach Euclid had originally adopted when he wrote the *Elements* and also because it would make Euclid's propositions more accessible to beginners, such as the students from the university of Tübingen, who, according to his experience, would get confused by the abundance and variations of letters.

tary of Euclid's *Elements* (based on the medieval Latin text of Campanus[1]), in his commentary on Prop. I.1, which teaches how to draw an equilateral triangle on a given line (Fig. 4, *a*).[2]

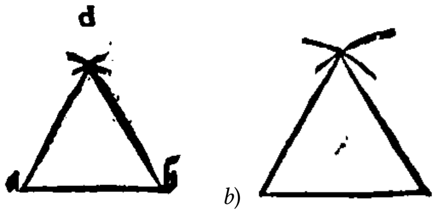
The translator. It must be noted that when one simply needs to describe the equilateral triangle on a given straight line, that is, when one does not need to demonstrate this operation, it is not necessary to describe the two circles entirely, but it merely suffices to draw the small part where they intersect in point D (as it appears in the second diagram) and to draw from the point D the two lines DA and DB and the triangle will be drawn. But when wanting to demonstrate and to determine the cause for which it is equilateral, it will be necessary to complete the two circles and to argue as was done above. The same thing is to be understood in many of the following problems.[3]

[1] Although Tartaglia also knew the translation of Zamberti and considered it philologically more accurate than Campanus' version of the *Elements*, he chose to base his Italian translation on the text of Campanus. One of the arguments he gave for follow Campanus rather than that of Zamberti for the translation of some of Euclid's technical terms was that his text proposed words that were more widespread and therefore more familiar to the common readers of Latin and vernacular texts (Tartaglia 1543, 6r).

[2] Tartaglia 1543, I.1, 15v: "Problema prima. Propositione prima. Possiamo sopra una data retta linea costituir un triangolo equilatero. Sia la data retta linea AB. voglio sopra di questa constituir uno triangolo equilatero. & per esequir tal cosa, io ponero il piede immobile del mio compasso, over sesto, sopra l'uno delle estremita della linea, cioe, in ponto A & l'altro piede mobile lo allargarò infino all'altra estremita, cioe, al ponto B & secondo la quantita di essa linea data per la terza petitione, descrivero il cerchio CBDF, dapoi questo di novo farò centro l'altra estremita di essa linea, cioe, il ponto B & per la medesima petitione (secondo la quantita della medesima linea) linearò il cerchio CADH, liquali cerchi se intersecaranno fra loro in duoi ponti, liqual sono C & D & l'uno de detti (poniamo il ponto D) continuarò con ambedue le estremita della data linea, tirando per la prima petitione le due linee DA & DB, et cosi sera constituido, il triangolo dab, ilqual dico esser equilatero: perche, dal ponto A ilqual è centro del cerchio CBDF sono tirate le linee AD & AB per infino alla circonferentia di quello, perilche seranno equale, per la diffinitione del cerchio, similmente anchora: perche, dal ponto B che è centro del cerchio CAD sono tirate le linee BA & BD per infino alla circonferentia di quello, quelle medesimamente seranno fra loro equale. Adonque perche l'una e l'altra delle due linee AD & BD è equale alla linea AB (come di sopra fu approvato) quelle medesime seranno anchora fra loro equale, per la prima concettione. Adonque sopra la data retta linea habbiamo collocato un triangolo equilatero, che è il preposito". Cf. Campanus 1482, a2v-a3r and Heath 1956, I, 241-242.

[3] Tartaglia 1543, Prop. I.1, 15v: "Il Tradottore: Bisogna notar che quando l'occorresse di descriver semplicemente il detto triangolo equilatero sopra una data retta linea, cioe, che'l non fusse di bisogno à far la demostratione di tal operare, non è necessario di descriver integralmente li detti

Figure 4



a) *b*)

In this context, Tartaglia presented the method to construct an equilateral triangle when it is not required to prove the geometrical validity of the construction, that is, to demonstrate that it properly allows to produce an equilateral triangle and that it has been carried out by operations authorised in the framework of Euclid's geometry, but only to effectively construct an equilateral triangle. As he explains then, this is done by merely drawing two intersecting arcs of circle according to the interval of the given line with a compass that maintains an opening equal to the length of the given line. This approach is properly practical insofar as it enables to know *how* to effectively obtain the requested figure without demonstrating *why* this construction is appropriate to this aim. As Tartaglia notes in conclusion, the same approach may be applied to many other Euclidean problems.

A similar discourse is also found in the 1562 German translation by the Heidelberg university professor Wilhelm Holtzmann (Xylander), in the commentary on Prop. I.1, after providing Euclid's original proof (Fig. 4, *b*).[1]

duoi cerchi, ma basta solamente à designar quella poca parte dove fanno la intersecatione in ponto D (come appare nella seconda figura) & dal detto ponto D tirar le due linee DA & DB & sera designato il detto triangolo: ma volendo dimostrar, & assignar la causa che quel sia equilatero egli e necessario à compire li detti duoi cerchi, & arguire come di sopra fu fatto: il medesimo si debbe intendere in molte delle sequente probleme".

[1] Although it is not the point here, it may be useful to note furthermore, regarding his treatment of Euclid's actual proof, even if Xylander's German version of the proposition is rather faithful to Euclid's text, at least more than the Latin text of Scheubel, in the sense that he follows the same construction procedure and offers a diagram and a lettering conform to the more classical treatment of the proposition, he nevertheless formulated Euclid's discourse in a more straightforward

Advice. May the beginners, with this, be advised and instructed that it is not necessary to trace out the two circles entirely, or to make them visible, but it suffices that you make with the compass two hidden traces according to the length of the given line, which go through each other crosswise. Then, what remains from such a circle is only useful to the demonstration. This is likewise to be understood in other propositions. What dimensions you should make an isosceles or a scalene triangle is taught by the following proposition 22, etc.[1]

In this commentary, which is entitled "Warnung" (which can be translated here as "Advice" or "Tip"), Xylander addresses the unlearned reader ("der ainfeltige"). The guidance offered is approximately the same as that which was provided by Tartaglia in the sense that it teaches how to perform the construction when no demonstration is needed. He also notes that this situation applies to other propositions of Euclid.

In Henry Billingsley's 1570 English translation and commentary of Euclid's *Elements*, which remained overall faithful to Euclid's text such as translated by Zamberti (even in comparison to Heath's modern translation),[2] most of the com-

and hands-on manner, using more common terms and referring to the use of the compass, which he uses, along with the observation of the diagram, as a means to empirically deduce the truthfulness of the construction. Xylander 1562, I.1, 6: "Wiewol *dise proposition leichtlich mag verstanden werden, auß beigesetzter figur*, will ich sich je doch (diewil sy die erst) weitleffig erkleren. (...) sollicher lini lenge begreiff ich mitt einem zirckel, unnd setz den ainen fuß in den puncten A und reiß mit dem andern den zirckel BCD darnach setz den ainen fuß in den puncten B unnd reiß den zirckel ACE *dise zwen zirckell werden on zweiffel gleich sein, dann sy baid mit onverruckhten zirckel, in ainer weittin beschribenn*" (my emphasis). Moreover, Xylander (*ibid.*) then clearly distinguished the construction from the proof, attributing to each of them a separate title: "Figur und Erklårung der ersten Proposition". And "Demonstratio, das ist, Grund und ursach diser Operation".

1 Xylander 1562, I.1, 7: "Warnung: Will hiemitt den ainfeltigen ermantt haben, unnd gewarnet. Das nit vonnötten die zwen zirckel gar außzureissen, oder die selben offenbar zumachen, sonder gnug ist, so du zwen verborgne riß, in der gebnen lini lenge, mit dem zirckel machest, die creutzweiß durch einander gehn. Dann das uberig von sollichem zirckel, dienet nur zur Demonstration, deßgleichen solt auch in andern verstehn. Welcher massen du ain gleichfussigen oder gar ungleichen triangell machen sollest, lehrt die 22 volgendt Propos. &c.".

2 Billingsley 1570, I.9, 18r: "The 4. Probleme. The 9. Proposition. To devide a rectiline angle geven, into two equall partes. Suppose that the rectiline angle geven be BAC. It is required to devide the angle BAC into two equal partes. In the line AB take a point at all adventures, & let the same be D. And (by the third proposition) from the lyne AC cutte of the line AE equall to AD. And (by the first peticion) draw a right line from the point D to the point E. And (by the first proposition) upon the line DE describe an equilater triangle and let the same be DFE, and (by the first peticion)

mentary on the problems of Book I are concluded by a separate and clearly delimited practical section, teaching for instance in Prop. I.9 how "to devide a rectiline angle into two equall parts Mechanically" (Table 5).[1]

As Billingsley writes at the very beginning, "mechanically" in this context is equivalent to "readily", which means "promptly" or "easily", but also without the demonstration. Billingsley thus aims to present a non-demonstrative but efficient and instrumental technique to perform the requested constructions, as did Tartaglia and Xylander.

As the latter, Billingsley makes explicit the use of the compass to perform the construction, which he had not done in his translation of Euclid's proof (contrary to Tartaglia and Xylander). The deictic words and sentences used here by Billingsley ("And here note"; "As in the figure here in the end of the other side put") are also marks of a practical discourse, since it allows him to address the reader directly and to contextualise his teaching, at least hypothetically and in the framework of the book. Such deictic sentences, which are not uncommon in commentaries of Euclid's *Elements*, held a crucial place in practical geometry treatises insofar as they often invited to understand or deduce the content of the teaching by simply looking at the diagram.

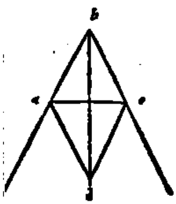
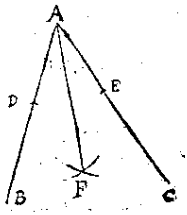
A similar approach is again proposed in the Latin commentary on Euclid's *Elements* by the Jesuit professor Christoph Clavius, first published in 1574. Clavius's treatment of Euclidean problems, also exemplified by Prop. I.9, is quite similar to that of Billingsley, insofar as he remains quite faithful to Euclid's proof[2]

drawe a right line from the poynte A to the point F. Then I say that the angle BAC is by ye line AF devided into two equal partes. For, forasmuch as AD is equall to AE, and AF is common to them both: therfore these two DA and AF, are equall to these two EA and AF, the one to the other. But (by the first proposition) the base DF is equall to the base EF: wherfore (by the 8. proposition) the angle DAF is equal to the angle FAE. Wherfore the rectiline angle geven, namely, BAC is devided into two equal partes by the right line AF, Which was required to be done". Cf. Zamberti 1505, a4v and Heath 1956, I, p. 264.

[1] In the commentary on Prop. I.9, this section follows several sections presenting related problems left aside by Euclid, such as the trisection of the angle (Billingsley 1570, 19r: "For to devide an acute angle into three equal partes, is (...) impossible: unles it be by the helpe of other lines which are of a mixt nature")., responses to objections (19r: "Here against this proposition may of the adversary be brought an instance. For he may cavill that the hed of the equilater triangle shall not fall betwene the two right lines, but in one of them, or without them both"). or alternative demonstrations (19r-v: "Divers cases in this proposition").

[2] Clavius 1611-1612, I.9, 35: "Problema 4. Propositio 9. Datum angulum rectilineum bifariam facere.

Table 5: Billingsley 1570, Prop. I.9.

<table>
<tr><th>Euclid, Prop. I.9</th><th>Billingsley, Prop. I.9 – Commentary, Part 4: Practical interpretation</th><th>Marginal notes</th></tr>
<tr><td></td><td>
</td><td></td></tr>
<tr><td>Let a point D be taken at random on AB; let AE be cut off from AC equal to AD; [I. 3] let DE be joined, and on DE let the equilateral triangle DEF be constructed; let AF be joined.[a]</td><td>This is to be noted, that if a man will mechanically or readily, not regardyng demonstration, devide the foresaid rectiline angle BAC, and so any other rectiline angle geven whatsoever, into two equall partes, he shall neede onely with one opening of the compasse taken at all adventures to marke the two pointes D and E, which cut of equal partes of the lin es AB and AC, howsoever they happen, and so making the centres the two points D and E, to describe two circles according to the openyng of the compasse: and from the point A to their intersection, which let be the point F to draw a right line: which shall devide the angle.</td><td>To devide a rectiline angle into two equall parts Mechanically.</td></tr>
</table>

[a] Euclid, Prop. I.9 (Heath 1956, I, 264).

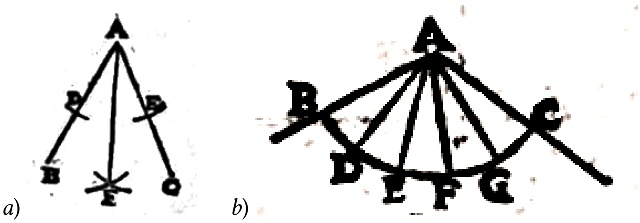


Figure 5

and was careful to provide a practical treatment of the proposition only in a clearly separate part. In this framework, the sections teaching how to perform the construction in practice are entitled *praxis* to explicitly mark their practical status and intention.

Practice. By what has been said, any rectilinear angle, such as BAC [Fig. 5, *a*], will be bisected quickly in the following manner: From the centre A, let the equal straight lines AD, AE be cut off with a compass according to any quantity. And with the compass remaining in the same position (you can change it however, if you want), let two arcs be drawn around the centre D and E which intersect in F. Therefore, the drawn straight line AF will bisect the angle BAC. If indeed the straight lines DF, EF have been led, these will be equal, as the radii of equal circles necessarily are. Wherefrom, as will have been demonstrated before, the angle DAF is equal to the angle EAF.[1]

Sit dividendus rectilineus angulus *bac*, bifariam, hoc est, in duos angulos æquales. In recta AB, sumatur quodcunque punctum D, & rectæ AD, secetur ex AC, recta AE, æqualis, ducaturque recta DE. Deinde super DE, constituatur triangulum æquilaterum DFE, & ducatur recta AF, dividens angulum BAC, in angulos BAF, CAF. Dico hos angulos inter se esse æquales. Cum enim latera DA, AF, trianguli DAF, æqualia sint lateribus EA, AF, trianguli EAF, utrumque utrique, quod DA, ipsi EA, per constructionem, sit æquale, & AF, commune; Sit autem & basis DF, basi EF, æqualis, propterea quod triangulum DFE, constructum est aequilaterum: Erit angulus DAF, angulo EAF, æqualis, ideoque angulus BAC divisus bifariam, quod erat faciendum". Cf. Commandino 1572, 12r and Heath 1956, I, 264.

[1] Clavius 1611-1612, I.9, 35: "*Praxis.* Dicto citius angulus quilibet rectilineus, ut BAC, bifariam secabitur, hoc modo. Ex centro A, circino aliquo abscindantur rectae aequales AD, AE, cuiuscunque magnitudinis. Et circino non variato (posses tamen ipsum variare, si velles) ex centris D, & E, describantur duo arcus secantes sese in F. Recta igitur ducta AF, secabit angulum BAC, bifariam. Si enim ducerentur rectae DF, EF, essent hae aequales, nempe semidiametri circulorum æqualium.

As in Billingsley's "mechanical" construction, Clavius' *praxis* is presented as a means to perform the requested construction in a quicker manner, which is again enabled by the use of the compass and by leaving aside the construction steps required for the demonstration, that is, the construction of the line *de* and of the equilateral triangle *def*.[1] During the enumeration of these different steps, Clavius also offers practical tips in parentheses, explaining directly to the reader that he can choose to adopt a different opening of the compass during the second step of the construction, if he wants, indicating that the construction of an isosceles triangle is as effective as that of an equilateral triangle to obtain the requested construction.

Now, if Clavius' intention in his *praxis* was to teach how to perform the construction in a quicker manner, he did not totally dismiss the demonstration, since he briefly referred to it when he mentioned the equality of the angles *daf* and *eaf* ("Wherefrom, as will have been demonstrated before, the angle DAF is equal to the angle EAF"), indicating that the practical construction is founded on the same principles as the original construction. He thus showed that, even if the lines necessary to Euclid's proof are not made visible in the practical construction, its various steps are nevertheless founded on the same principles as the original construction.

Nevertheless, Clavius expressed reservations concerning the intelligibility of Euclid's construction, and of the correlated diagram, in comparison with those accomplished by the means of "pure practice" ("nuda praxis"), insofar as they would generate confusion rather than clarity due to the great number of lines that make up Euclid's diagram.

unde ut prius demonstrabitur, angulum DAF, aequalem esse angulo EAF".

[1] Raynaud (Raynaud 2015, 11) interprets Clavius' *praxes* as *exempla*, in the sense of application of the reasoning to a particular problem. Lee, on the other hand, wrote on these sections of Clavius' commentary: "the establishment of a praxis section can be interpreted as an attempt to assemble the tradition of transcribed diagrams in manuscripts and the new trend of tool-based diagrams in printed editions" (Lee 2018, 18). I think that this notion has a larger scope than what is suggested by either of these definitions, aiming to represent a section in which Clavius taught a version of Euclid's construction that is focused on the means to obtain it, that is, the operations required to produce it concretely and effectively, rather than on the causes of its accuracy, similarly to how it was taught by Tartaglia, Xylander and Billingsley. Its practical character, which is acknowledged in the title, is marked by its emphasis on doing over knowing, which does not however mean that the taught construction is not mathematically valid, as I will show further.

But we have not described the above-mentioned lines, so that pure practice could be used, which we will observe in the other practices as much as it is possible, so that the multitude of the lines does not set us in the dark and create confusion.[1]

Therefore, the *praxis* for Clavius would not only serve as a means to perform the construction more promptly, but would also make the proposition easier to understand. In the scholium that follows this practical section, in which he started by mentioning the complex and mechanically generated curves through which ancient mathematicians proceeded to solve the famous problem of the trisection of the angle (as had Billingsley in his commentary),[2] Clavius also briefly taught the manner in which an unlearned person would divide a given rectilinear angle in any number of equal angles.

But meanwhile, if someone wanted to divide any given rectilinear angle in any number of equal parts in the manner of the unlearned, as could be said, it will be necessary for them to use a compass, so that by making an attempt and by repeating this practice several times, they will thereby surely reach the desired aim by this means. Let the angle BAC [Fig. 5, *b*] be a rectilinear angle which is to be divided in 5 equal angles. From A taken as a centre, let the arc of circle BC be described with any interval, cutting the straight lines AB, AC, in B, and C. Then let this arc be divided by the means of the compass (sometimes opening, sometimes closing its legs further until they have the required distance) in as many equal parts as the proposed angle is to be divided in, as in five parts in the points D, E, F, G in the provided example.[3]

According to this manner, Clavius advised the reader to use a compass whose

[1] Clavius 1611-1612, 35: "Non descripsimus autem dictas lineas, ut nuda praxis haberetur: Id quod in aliis quoque praxibus, quoad eius fieri poterit, observabimus, ne linearum multitudo tenebras nobis offundat, pariatque confusionem".
[2] Billingsley 1570, 19r: "It is impossible to devide an acute rectiline angle into three equall partes without the helpe of lines which are of a mixt nature".
[3] Clavius 1611-1612, I.9, 35: "Interim vero, si quis angulum rectilineum quemcunque propositum in quotvis partes aequales dividere desideret rudi, ut dicitur, Minerva, uti eum necesse erit circino, ut quasi attentando, & saepius repetendo praxim ipsam, ad finem desideratum perveniat, hac nimirum ratione. Sit angulus rectilineus BAC, dividendus in 5 angulos æquales. Ex A, centro describatur arcus circuli BC, ad quodcunque intervallum, secans rectas AB, AC, in B, & C. Deinde hic arcus beneficio circini (eius crura modo dilatando magis, modo restringendo, donec debitam habeant distantiam) dividatur in tot partes æquales, in quot angulus propositus est dividendus, ut in exemplo proposito in quinque partes in punctis D, E, F, G".

opening is increased or decreased until the needed interval is obtained so that the angle be approximately divided in the desired number of equal parts, proceeding therefore by trial and error.

Hence, the teaching provided in the last examples show that "practical" here is not merely related to the use of instruments and to the fact of performing a construction concretely, but also the fact of leaving aside the demonstration (even if it underlies a given practice) to focus on the gestures and operations to perform to concretely obtain the sought figure, notably in order to save time. In this sense, this construction, just as the practical construction presented before by Clavius and those taught by Tartaglia, Xylander and Billingsley evoke the notion of "constructive geometry" which L. Shelby used to describe the teaching of medieval master masons, and which is never "mathematically demonstrated to be correct".[1]

2.3. Empirical approaches to Euclid's theorems

Given their speculative finality,[2] theorems did not invite the translators and commentators of Euclid to apply a practical approach as much as problems, which is notably why we can only rarely find theorems in practical geometry treatises, as opposed to problems.[3] And when commentators of Euclid did propose a more practical interpretation of a given theorem, the practical character

[1] Shelby 1972, 413.

[2] Proclus (Friedlein 1873, 77; transl. Morrow 1992, 63): "The propositions that follow from the first principles he divides into problems and theorems, (...) *the latter concerned with demonstrating inherent properties belonging to each figure*".

[3] Notable exceptions are Recorde and Digges. But while Digges, who only quotes a handfull of theorems, merely provides the enunciations, Recorde provides proofs for the theorems (for Books I-IV). These proofs, which are called examples, are however not those found in the classical Euclidean text, but mostly practical proofs, using numbers for instance (as in Prop. I.4 or *The first Theoreme*, c1r-v), or purely descriptive proofs, pointing to the parts of the diagram, referring to Euclid for the proper demonstration of the proposition. E.g. Recorde 1551, Prop. I.5, c1v-c2r: "*The second Theoreme.* In twileke triangles the ij corners that be about the ground line, are equal togither. And if the sides that be equal, be drawen out in length then wil the corners that are under the ground line, be equal also togither. Example. ABC is a twileke triangle, for the one side AC, is equal to the other side BC. And therfore I saye that the inner corners A and B, which are about the ground lines, (that is AB) be equall togither. And farther if CA and CB bee drawen forthe unto D and E, as you se that I have drawen them, then saye I that the two utter angles under A and B are equal also togither: as

of the chosen approach is not as evident as in the case of problems. In this context, the practical treatment of theorems mainly consisted in the addition of empirical arguments (notably by visual means) or in the use of computations, which would offer a more immediate or "hands-on" means of verification of the demonstrated quantitative relation between figures or lines.[1] In this framework, an empirical approach to Euclid's theorems appears, for instance, when dealing with Prop. I.4, which proves the congruence of two triangles by superposition,[2] and which Peletier (as mentioned above) had rejected as a mechanical and non-geometrical mode of demonstration in his commentary on the *Elements*.[3] Peletier's rejection was due to the fact that he interpreted it as a constructive procedure (instead of a hypothetical reasoning) and that it would therefore not be legitimated by any constructive postulate, relying solely on an empirical mode of assessment of the congruence of geometrical figures. Although Prop.

the theorem said. *The profe wherof, as of al the rest, shal apeare in Euclide*, whome I intende to set foorth in english with sondry new additions, if I may perceaue that it wilbe thankfully taken" (my emphasis).

[1] The fact of performing a computation, as determining the area of a given rectilinear figure from the knowledge of the quantities of its sides, may indeed be regarded as a practical and empirical means of verification of a given theorem enunciating the quantitative relation between two figures (e.g. the equality of triangles with equal bases and in the same parallels in Prop. I.38), since readers find themselves able to mentally manipulate the considered geometrical objects by operating on the quantities of their parts, following a set of practical rules most often taught in treatises of practical arithmetic. By comparison, the fact of following a demonstration based on prior propositions or principles and on rules of logical deductions, even if possessing a practical character based on the application of logical precepts, would correspond, for the reader of Euclid's *Elements*, to a mostly passive and purely rational means of verification of the truth of the theorem. The properly practical and empirical character of the use of computations in this context is asserted by Tartaglia (in Tartaglia 1543, Df. V.9, 64v: "spesse volte il studente che vede con la esperientia de numeri verificarse la propositione preposta, non si cura di intendere quella per demostratione") and Xylander (in Xylander 1562, I.35, 22: "so du die warhait und gwiß diser und volgender prop. durch rechnung in ealen erfaren wo̊lltest (wellichs seer nutʒlich zu dem das es kurtʒweilig und lustig ist").

[2] Euclid, Prop. I.4 (Heath 1956, I, 247): "If two triangles have the two sides equal to two sides respectively, and have the angles contained by the equal straight lines equal, they will also have the base equal to the base, the triangle will be equal to the triangle, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend".

[3] Peletier, *Euclidis elementa,* 1557, I.4: "Figuras Figuris superponere, Mechanicum quippiam esse: intelligere verò, id demum esse Mathematicum". On Peletier's rejection of superposition and the early modern debate on Euclid's use of superposition, see Mancosu (1996, 29-31), Loget (2000, 171-177), Palmieri (2009, 474-476) and Axworthy (2018).

I.4 gave rise to different types of practical treatments, I will here only focus on the cases of Scheubel and Xylander, as they most conspicuously manifest an empirical approach to this proposition.

After presenting the exposition and specification of Prop. I.4, which state what is given and what is to be demonstrated,[1] Scheubel wrote that a visual demonstration of this proposition (*ocularis quaedam demonstratio*) is admissible and even ought to be admitted, since "the thing appears to the sense nearly as it is".

A visual demonstration of this thing ought to be admitted at this point. But if someone wanted to immediately deny this, because the thing appears to the sense nearly as it is, and is evident insofar as it is true and known to all, it follows that he will finally be compelled to admit the opposite, That two straight line comprise a space, by a reduction to absurdity.[2]

Although Scheubel provides at this stage the more classical diagram of two identical triangles placed side by side (only without the lettering[3]), the assertion of the possibility to prove this proposition through a visual demonstration is connected to the diagram Scheubel provides towards the end of the proposition (Fig. 6, *a*). This diagram aims to visually demonstrate the impossibility of the case when the bases of two superposed triangles are not congruent although the two remaining sides and the contained angle are mutually equal.

[1] Euclid, Prop. I.4 (Heath 1956, I, 247): "Let ABC, DEF be two triangles having the two sides AB, AC equal to the two sides DE, DF respectively, namely AB to DE and AC to DF, and the angle BAC equal to the angle EDF. I say that the base BC is also equal to the base EF, the triangle ABC will be equal to the triangle DEF, and the remaining angles will be equal to the remaining angles respectively, namely those which the equal sides subtend, that is, the angle ABC to the angle DEF, and the angle ACB to the angle DFE". Cf. Scheubel 1550, Prop. I.4, 85: "Praescribantur huiusmodi duo triangula, qualia haec propositio requirit, quorum nimirum unius duo latera, duobus lateribus alterius æqualia sint: atque angulus deinde sub aequalibus lateribus unius, angulo sub aequalibus trianguli alterius comprehenso aequalis: dico quod & horum triangulorum bases, ipsa quoque triangula tota, atque reliqui anguli reliquis angulis utrinque inter se aequales sint".

[2] Scheubel 1550, Prop. I.4, 85: "Huius rei nunc accedere deberet ocularis quaedam demonstratio: sed quia ad sensum quasi ita sese habere res apparet, & evidens est, tanquam vera atque omnibus nota relinquitur, cum statim, hoc si quis negare velit, oppositum eius, ad extremum, Quod duae rectae spacium compraehendant, ut sequitur, fateri cogatur, reductione ad absurdum".

[3] As said, Scheubel systematically suppressed the lettering that is applied to Euclid's diagrams in the more classical editions.

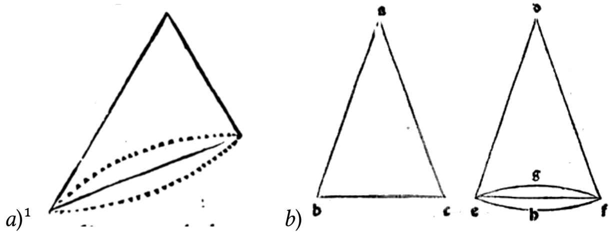


Figure 6: *a*) Scheubel 1550, I.4, p. 86; *b*) Campanus 1482, I.4, a3v.

This diagram actually has the same function as the more classical diagram of Prop. I.4 found for instance in Campanus (Fig. 6, *b*). However, the dotted lines aim to properly display at a glance the impossibility of the hypothesis envisaged in Euclid's proof.

In order to facilitate the reading of the diagram, Scheubel also explains, in his *Admonitio*, the function of the drawn dots, and of the dotted lines thereby,[2] in the interpretation of Euclid's *reductio ad absurdum*.

Notice. The argument leading to absurdity is represented by points in the figures, since he who does not easily concede this to be true will finally be convinced by some reduction to impossibility, so that he will somehow, through the repugnance to absurdity, withdraw in favor of the confession of the truth, which the readers will find here, as also in other places, through the drawing of points.[3]

Hence, Scheubel's approach to Euclid's theorem can be considered practical in the sense that it heavily relied on visual inference to persuade readers of the truth of the theorem, inviting them to mentally reconstruct the steps of the demonstration by means of the diagram and thus to gain a first-hand experience of its truth. It was indeed common, in practical geometry treatises to

[2] On the use and function of dotted lines in the sixteenth-century Euclidean tradition, see Lee 2018 and Lee 2020, 520-524.

[3] Scheubel 1550, I.4, 85-86: "Admonitio. Per puncta in figuris, representatur ratio ducens ad absurdum, ut qui facilis non esset in concedendo id quod verum est, tandem convincatur reductione quadam ad impossibile, ut hac offensione absurditatis quodammodo resiliat ad confessionem veri. Quod ut hoc loco, ita etiam alijs locis à me factum reperient Lectores, designatione punctorum".

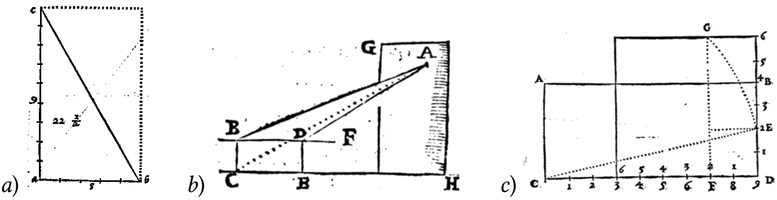


Figure 7: *a*) Bullant, *Geometrie et horologiographie pratique*, 1562; *b*) Merliers, *La practique de géométrie*, 1575; *c* Clavius, *Geometria practica*, 1604.

simply point to the diagram and invite the reader to proceed from there with the deduction of the proposition rather than to offer a formal demonstration of a proposition, as in Bovelles's *Geometrie practique*.[1]

Moreover, Scheubel's appeal to dotted lines introduced a kind of diagram that was often found in sixteenth-century practical geometry treatises (Figs. 7, *a-c*).[2]

Even if Scheubel did also provide the textual demonstration *ad absurdum* found in Euclid's original text, the fact that he asserted here the value of the visual demonstration and of sensed evidence contrasts with the representation of Euclidean mathematical demonstrations as purely rational, as was presented, for instance, in Foix-Candale's commentary on Euclid.[3]

[1] Bovelles 1547, I.26, 11r : "une ligne droicte ne peust toucher un cercle sur deux poincts: mais sur un seul. Ce est assez evident pat tout, & se peust facilement entendre par les figures cy devant descriptes". and I.27, 11r: "Si deux cercles touchent l'un l'autre, ce sera sur les seul poinct: sur lequel une mesme droicte ligne un peust toucher tous deux. Regarde la presente figure, & clerement entendras le propos. Car la ligne ABC, touche deux cercles sur un mesme poinct, sur lequel pareillement lesdicts cercles touchent l'un l'autre, sans soy diviser aucunement, & sans copper ladicte ligne ABC" (my emphasis).

[2] Admittedly, these examples were published later than Scheubel's Euclid, but it shows how common it was to find such diagrams in practical geometry treatises. Moreover, the sixteenth-century commentators or editors of Euclid's *Elements* that made use of dotted lines are among those who adopted the most practical approach to Euclid's propositions, such as Xylander (e.g. I.7, 9), Pierre Forcadel (e.g. I.43, 36v), Billingsley (e.g. Df. VI.6, 155r), Clavius (e.g. I.11, 36-37) and Dou (e.g. I.5, 4).

[3] Foix-Candale, *Euclidis elementa*, 1566, I.4, 5v: "Nam Campanus ac Theon hanc demonstrantes, triangulum triangulo superponunt, angulumque angulo, sive latus lateri, demonstrationem potius *instrumento palpantes, quàm ratione firmantes*: quod tanquam *prorsus alienum à vero disciplinarum cultu* reiicientes". See *supra*, §1.9.

Xylander's treatment of Prop. I.4 is quite similar to Scheubel's in the sense that, after providing the exposition and specification, he stated that this theorem, if it can be proved and demonstrated, should be considered as evident.

This proposition concerns in general a property of the triangle, when two triangles are held against each other and compared. (…) Now in order to understand this proposition, take the two triangles A and B drawn here on the side. The triangle A has two lines or sides equal to the two lines of the triangle B, namely, the line AB is equal to the line DE, and BC to the line EF, also the angle ABC (…) is equal to the angle DEF, that is, to the angle on which stands the letter E. Therefore, from this, it is certain that the line AC is also equal to the line DF and that the angle BCA is equal to the angle EFD. Then, they will be, as shown, subtended by equal lines, as AB and DE. The angle BAC (however big it is), is also contained by lines equal to those which contain the angle EDF, equal to it. From this, it follows finally that the whole triangle A must to be equal to the whole triangle B, which should not only be considered as evident (*augenscheinlich*), but may be proved and demonstrated. I leave this out. The cause is indicated above.[1]

In this passage, Xylander seems to take for granted that Prop. I.4 is evident, since as he writes towards the end that, in addition to the fact that it is evident, this proposition may also be proved and demonstrated, which is what Euclid did.

The German adverb *augenscheinlich* used here by Xylander fully conveys the Latin notion of *evidentia*, as what is plainly visible (from *ex + video*), but it does so in a properly concrete manner, specifically pointing to the vision of the eye (*auge*), instead of vision in general, which could be ocular or intellectual. The empirical character of evidence in this context is supported by the fact that, in

[1] Xylander 1562, I.4, 8: " Dise proposition begreifft in gemain ain aigenschafft des triangels, so zwen gegen einander gehalten und vergleicht werden. (…) Nun zuverstehn dise proposition, Nim fur dich die zwen hieneben verzaichneten triangel A und B, der triangel A hat zwo linien oder seitten gleich den zwo linien des triangells B, nemlich die lini AB ist gleich der lini DE, und BC der lini EF, auch der winckel ABC (…) ist gleich dem winckel DEF, das ist dem winckel darob der buchstab E steht. Derhalben wa dem also, so ist gewiß, das auch die lini AC, der lini DF gleich ist, und der winckel BCA dem winckel EFD gleich, dann inen werden, wie angezaigt, gleiche linien underzogen, als AB und DE, sy seind auch (welches eben so vil ist) mit gleichen linien begriffen des gleichen der winckel BAC, dem winckel EDF gleich. Darauß entlich volgt, das der gantʒ triangel A, dem gantʒen triangel B gleich muß sein, welches nit allain augenscheinlich zu erachten ist, sonder auch mag erwisen und demonstriert werden. Laß ich auß, ursach ist oben angezaigt" (my emphasis).

the early modern German language, as was notably demonstrated by T. Morel,[1] the notion of *Augenschein* was used to describe direct ocular inspection, as in the context of subterranean cartography, that is, when depicting the disposition of ore veins in mining pits, as well as the borders between concessions within mines, through which were established *Augenscheinkarten* (i.e. "maps by visual inspection"). In this context, as in the establishment of other types of cartographical representations, visual inspection had a properly legal value, being considered as a reliable testimony in the case of disputes relating to property.[2] J. Dumasy-Rabineau showed that ocular inspection and the diagrams drawn on the basis of the testimony of the eye were also regarded as proper witnesses and evidence in the legal sense in the medieval and Renaissance French practice of law.[3] In this context, the "vues figurées" (i.e. "depicted ocular inspection"), which were related to geometrical proofs by the means of diagrams as used in Bartolo de Sassoferrato's fourteenth-century treatise *Tiberiadis* or *De fluminibus*,[4] could thus be taken as a "proofs by vision".

The concrete character of the evidence (or *Augenscheinlichkeit*) in Xylander's commentary on Prop. I.4 is corroborated by his introductory remarks, in which the proposition is presented as dealing with a property of the triangle that allows two triangles to be "held against each other and compared" ("gegen einander gehalten und vergleicht").[5] According to this passage, Xylander appears to admit that Euclid's demonstration of the congruence of triangles necessarily implies the actual, mechanically performed, superposition of the two triangles, which is precisely how Peletier understood this proposition and what incited him to reject its mode of demonstration. The evidence of this proposition would thus be gained, for Xylander, from empirical judgment in the sense that the fact of mechanically holding two plane figures against each other, even in the imagination, would allow for an immediate and undeniable apprehension of the con-

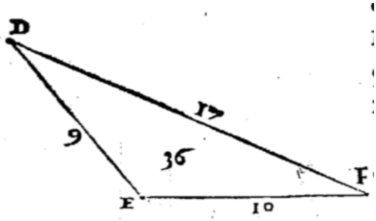
[1] Morel 2022, chapter 5 (p. 151-156). I thank T. Morel for having made the relevant part of his forthcoming book available to me.

[2] On "forensic cartography" by *Augenschein* in medieval and early modern Germany, see Horst 2014.

[3] Dumasy-Rabineau 2013.

[4] Dumasy-Rabineau 2013, 815. See also Frova 1999.

[5] Xylander 1562, I.4, 8: "Erklärung: Dise proposition begreifft in gemain ain aigenschafft des triangels, so zwen gegen einander gehalten und vergleicht werden".

a) 

b) 

Figure 8: *a*) Oronce Fine, *Geometria practica*, 1532; *b*) Jean Errard, *Geometrie et practique d'icelle*, 1594.

gruence, and thus of the equality, of the two compared figures. Moreover, Xylander does not provide any demonstration at all, stating that he left it out ("laß ich auß") because "the cause was indicated above" ("ursach is oben angezaigt"). Now, it was common in Xylander's Euclid, as shown in his commentary of various other propositions of the *Elements*, to simply invite the reader to draw the truth of the proposition from the drawn figure, as in Prop. I.24.[1]

2.4. Numerical treatment of Euclidean propositions

As mentioned earlier, the adoption of a numerical treatment of magnitudes, which was typical of practical geometry treatises (as central to the art of measuring and to the practice of surveyors) (Figs. 8, *a*-*b*), occurred more and more often throughout the early modern Euclidean tradition.[2]

As said, this approach to magnitudes was intrinsically non-Euclidean, since numbers and magnitudes were dealt with separately, even when dealing with the principles and properties of ratios and proportions which are applicable to both types of quantity. For this reason, already in the medieval tradition,

[1] Xylander 1562, I.24, 17: "So zwen triangel haben je ainer zwo seitten des andern zwo seitten gleich, aber der ain hatt ein gr̊ossern winckel mitt disen gleichen seitten begriffen, der hat auch ain grosser basim. Erklårung. *In dise figur hastu den gantzen handel augenscheinlich*, dann *besich* die zwen triangel BAC und DAE" (my emphasis).

[2] On this topic, see Malet 2006 and Malet 2012. See also Corry 2022.

in addition to the fact that specific numerical values were introduced in the arithmetical books instead of, or in correlation with, the straight lines that were used in the Greek tradition to express numbers in a general manner,[1] numbers were sometimes introduced in Book V, which deals with the theory of ratios and proportions applied to magnitude.[2] In the sixteenth-century printed tradition, the introduction of numerical examples in the propositions of Book V was very common.[3]

Book II, which mostly deals with equivalences between areas of quadrangles, also easily invited an arithmetical and even algebraic treatment of Euclid's geometrical propositions, for which commentators such as Billingsley and Clavius included in their commentaries the arithmetical rewriting of the ten first propositions of Book II by Barlaam of Seminara.[4]

[1] This is itself intrinsically practical in the sense that numbers are dealt with in a less abstract and general manner, which would notably enable the readers to verify the propositions in an operative manner. The fact of providing specific numbers is related in this respect to the attribution of specific lengths, areas and volumes to abstract geometrical magnitudes.

[2] See the thirteenth-century adaptation of Robert of Chester's compilation edited in Busard 1996, p.172-173; the commentary of Adelard's translation by Johannes de Tinemue (Adelard III) in Busard 2001, 127-128 and the medieval text of Campanus' commentary in Busard 2005, 168-170.

[3] Among those who presented numerical examples in Book V, at least within the diagrams, are Fine (Df. V.6, 110-111); Scheubel (e.g. Scheubel 1550, Fg. V.6, 129: *Exempla in numeris sunt* or V.3, 238: *Exemplum in numeris*), Xylander (e.g. Xylander 1562, Df. V.8: 125), Foix-Candale (e.g. Foix-Candale 1566, Df. V.6, 40r), Billingsley (e.g. Billingsley 1570, Df. V.1, 126r), Commandino (e.g. Commandino 1572, Df. V.5: 58r), Clavius (e.g. Clavius 1611-1612, Df. V.2: 167) and Élie Vinet (e.g. Vinet 1575, Df. V.5, b4r-v) and Jan Pieterszoon Dou (Dou 1606, V.1, 119).

[4] Billingsley 1570, Prop. II.1: "Because that all the Propositions of this second booke for the most part are true both in lines and in numbers, and may be declared by both: therefore have I have added to every Proposition convenient numbers for the manifestation of the same. And to the end the studious and diligent reader may the more fully perceave and understand the agrement of this art of Geometry with the science of Arithmetique, and how nere & deare sisters they are together, so that the one cannot without great blemish be without the other, I have here also ioyned a little booke of Arithmetique written by one *Barlaam*, a Greeke authour a man of greate knowledge. In whiche booke are by the authour demonstrated many of the selfe same proprieties and passions in number, which Euclide in this his second boke hath demonstrated in magnitude, namely, the first ten propositions as they follow in order. Which is undoubtedly great pleasure to consider, also great increase & furniture of knowledge. Whose Propositions are set orderly after the propositions of Euclide, every one of *Barlaam* correspondent to the same of Euclide". and Clavius 1611-1612, 367: "Quoniam ad theorema sequens demonstrandum Theon quaedam assumit in numeris, quae demonstrata sunt de lineis libro secundo, perinde ac si eadem de numeris essent ostensa; non alienum instituto nostro

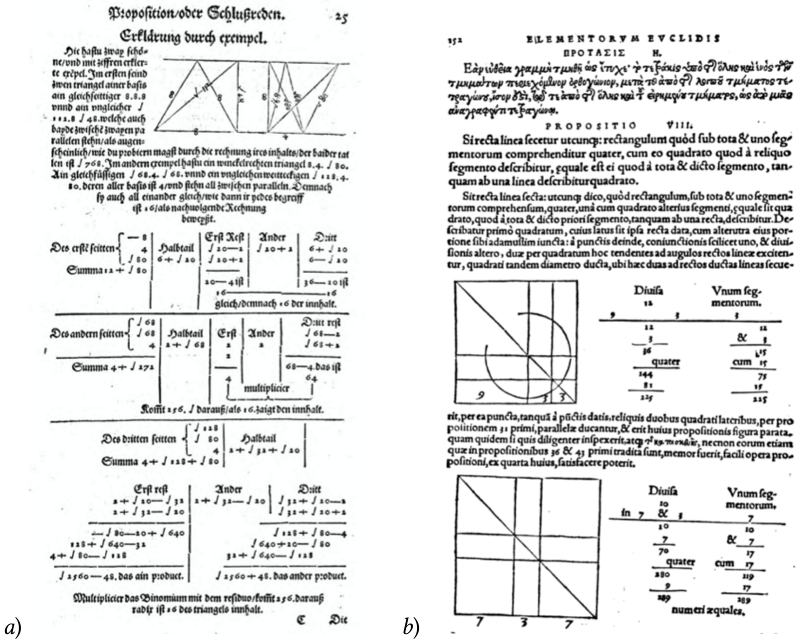


Figure 9: *a*) Scheubel 1550, II.8, 152; *b*) Xylander 1562, I.37, 25.

Scheubel and Xylander also manifested the connection between discrete and continuous quantity by adding a number of sections (mostly in books I, II and VI) which present an arithmetical and algebraic treatment of Euclid's geometrical propositions in the form of annexed computations (Figs. 9, *a-b*).

References to specific numerical values in Euclid's other geometrical books were also punctually made in various other commentaries, such as those by Peletier[1] and Forcadel,[2] where we notably find specific units of measures.

duximus, nonnulla ex ijs, quae Geometrice ab Euclide libro 2 demonstrata sunt de lineis, hoc loco de numeris demonstrare, Quod idem & *Barlaam monachum* fecisse à nonnullis est traditum. Sequemur autem eundem ordinem, quem Euclidem in secundo libro tenuisse conspicimus". On Barlaam of Seminara and his reception in the early modern era, see Corry 2022. See also, for the arithmetical treatment of Book II, Pacioli 1509, II.9, 14r: "Illi trianguli sunt similes 5 ADF & ACP & ideo laterum proportionalium per 4 sexti, quia angulus D maioris & angulus C minoris sunt recti & angulus A unius est idem cum angulo A alterius sequitur per 32 primi angulos P parvi & F magni esse equales & sic latera illos continentia sunt proportionalia per dictam 4 sexti & ideo ponendo AD 9 & DF 3 erit CP 2, PG 1, AP ℜ40, quia AC 6 & PF ℜ10 cetera sunt clara & pratice dicitur vulgariter 'se AD basa del grande mida DF catecto che midara AC basa del picolo, cioe se 9 mida 3 che midara 6' operando habebis ut iam diximus" (my emphasis).

[1] Peletier 1557, I.47, 49r: "Nunc autem hoc Theorema quonam pacto ad Numeros accommodetur, obiter ostendemus. In Numeris itaque locum praecipuè habet, quum maximus ad medium fuerit ut 5 ad 4: scilicet in proportione, quam vocant, sesquitertia. Eiusmodi sunt 3, 4, 5: tresque 6, 8, 10: & 12, 16, 20: sicque continuo progressu. Quadratum enim 20 est 400, ac tantundem efficiunt 144 cum 256, quae sunt Quadrata ex 12 & 16. Sed & cognito minimo laterum in Numeris, Scalenum Rectangulum sic absolves. Dimidium cogniti duc in se: à producto aufert unitatem: habebis alterum latus. Huic adde binarium: fiet maximum latus, seu subtensa. ut, sit minimum latus 10: Horum dimidium duc in se: fiunt 25: à quibus ablata unitate, supersunt 24, medium latus: his adde binarium: fiunt 26, subtensa. Horum enim Quadratum 676: Et tantundem efficiunt 100 & 576, Quadrata ex 10 & 24".

[2] Forcadel 1564, I.36, 31v: "Par ceste proposition aussi, si on nous dict qu'un parallelogramme contient 486 parallelogrammes esgaux entr'eux, & qu'il est une fois & demy autant long que large: Nous pourrons prendre pour les deux costez d'icelly 3 & 2 lesquels multipliez ensemble dont, 6 qui sont esgaux ou valent 486, par cest proposition & par la premier commune sentence, & par ainsi l'un des 6 en vaudra 81 des autres, duquel la racine quarree qui est 9 multipliee par 3 & par 2 faict 27 & 18 pour les deux costez du parallelogramme. Et si le rectangle contenoit *30 pieds quarrez*, certainement l'un des six en contiendroient, 5, & les quarrez des deux costez en contiendroient 45 & 20: & par ainsi les deux costez seroient racine de 45, & racine de 20". Specific units of measure are also found in Clavius' commentary on Book II, in Clavius (1611-1612, I, 82-83), Df. II.1: "Habet autem comprehensio haec parallelogrammi rectanguli sub duabus rectis lineis angulum rectum continentibus, magnam affinitatem cum multiplicatione unius numeri in alterum. Sicut enim ex multiplicatione 3 in 4 producitur numerus 12 qui in formam parallelogrammi constituitur, unde & contineri dicitur sub 3 & 4. Ita quoque parallelogrammum ABCD, comprehensum sub duabus rectis

Scheubel, in particular, introduced his commentary on Euclid with a nearly eighty-page treatise of algebra (*Brevis regularum algebrae descriptio, una cum demonstrationibus geometricis*), in which he taught the basic notions of algebra and rules of algebraic computations, using concrete examples from money-changing, commercial contracts, military art and the measuring of concrete objects.[1]

As an example of this numerical approach to Euclid' geometry, I will only consider here Scheubel's commentary on Prop. I.34,[2] in which the arithmetical treatment of the proposition is provided immediately after the proof[3] within a separate part entitled *Appendix*. In many cases, such sections were designated by Scheubel *numerorum praxis*[4] or simply *praxis*,[5] using the terminology which Clavius would later use for his practical treatment of Euclid's constructions in Book I.[6]

As Scheubel explains here, the geometrical propositions which deal with triangular areas, such as I.34, can be easily applied to numbers and treated arithmetically. His aim is thus to teach a general method to determine the areas of any triangle from the knowledge of their sides.

Appendix. Since this thirty-fourth proposition, and also many of those that follow, are

AB, BC, quarum illa sit *3 palmorum*, haec autem 4 constat *12 palmis* quadratis, qui quidem ex ductu lineae AB, 3 palmorum in lineam BC, *4 palmorum* producuntur" (my emphasis).

[1] Scheubel 1550, 3: "Proinde harum regularum exempla, cum eodem modo, quo in communi negociatione alias monetarum, mensurarum & ponderum, atque etiam quarumlibet aliarum rerum numeri, enuncientur, his duobus exemplis positis, puto iam facile omne pro positum exemplum exprimi posse, quare de enunciatione iam satis".; 19: "4. Großus valet 10 nummulis, 24 verò großi florinum constituunt. Aliquis nunc florinum permutans, tot pro eo grossos, quot nummulos cupiens, quaeritur quantum utriusque recipiat. (...) 5. Est area quaedem quadrangulis, continens in superficie 588 areolas, inter se & toti similes. Quia autem huius areae longitudo ad latitudinem est, ut 4 ad 3: quanta ipsius longitudo, latitudo item sit, quaeritur. (...) 6. Dux in castris suo sub imperio habet aliquot mille milites".

[2] Euclid, I.34 (Heath 1956, I, 323): "In parallelogrammic areas the opposite sides and angles are equal to one another, and the diameter bisects the areas".

[3] In this context, the text of the proposition is already accompanied with numerically-determined diagrams.

[4] Scheubel 1550, I.40, 120 and I.41, 121.

[5] Scheubel 1550, II.4, 146.

[6] See also Scheubel 1550, I.36, 117 or I.37, 118: *Nunc quantum ad praxim numerorum*; II.4, 143: *Sequitur calculus* or 144 *Aequatio*.

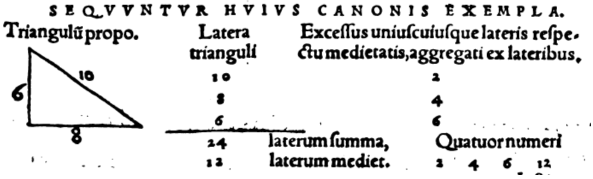


Figure 10: Scheubel 1550, I.34, 110.

found to be true for numbers, that is, for discrete quantity, as much as for continuous quantity, by which we may therefore appropriately teach a general rule by which the areas of any type of triangles (provided their sides are known) can be found, it was necessary to provide it below through these words:
Let first the sides of the triangle, whose area it is proposed to find, be added together, then let each of the sides of the triangle be subtracted from the half of this result. Three numbers will remain, which, if they are multiplied with each other, along with the half of the sides added together as a fourth number, that is, the first with the second, their product with the third, and what will then be produced with the fourth number (and the order according to which the numbers are taken, namely as the first, the second, the third and the fourth, does not convey the order according to which they should be computed), then through the square root of this last product will be exhibited how great the area of the proposed triangle will be.[1]

The method taught here in general terms by Scheubel is then applied to a specific case (Fig. 10). Using the numerical values given in this example, the

[1] Scheubel 1550, I.34, 110: "Appendix. Quoniam autem haec propositio 34, & multae etiam sequentes, in numeris, quantitate nimirum discreta, non minus atque in quantitate continua, veræ esse reperiuntur, quo id ostendamus commodius, canonem quendam generalem, per quem omnis generis triangulorum (modo latera eorum nota fuerint) areæ inveniri possent, subijcere necesse fuit, his verbis. Trianguli, cuius aream propositum est invenire, latera primò in unum colligantur, à medietate deinde huius collecti singula trianguli latera subtrahantur. Relinquuntur autem tres numeri, qui unà cum medietate collecti ex lateribus, tanquam numero quarto, si inter se multiplicati fuerint, primus scilicet cum secundo productum hoc cum tertio, quodque iam producetur cum numero quarto (nec refert quo ordine numeri sumantur, quive pro primo, secundo, tertio vel quarto reputetur) tum huius ultimi producti radice quadrata, quanta propositi trianguli area fuerit, manifestabitur".

EXEMPLVM CANONIS ALIVD.

Eſt autem in hac 34 propoſitione triangulum figuræ primæ.

Latera	Exceſſus
$\sqrt{72}$	$6 - \sqrt{18}$
6	$\sqrt{18}$
6	$\sqrt{18}$
Sum. $12 + \sqrt{72}$	Meditas $6 + \sqrt{18}$.

Figure 11: Scheubel 1550, I.34, 111.

method Scheubel taught to obtain the area of the considered triangle may be summed up as follows:

1. Add together the lengths of the three sides of the triangle, which are in the given example: 10, 8, 6. Added together, these are equal to 24.
2. Divide the result in half, from which is obtained a fourth number (here called *medietas*): 24 : 2 = 12.
3. Subtract the amount of each sides from the *medietas*: 12 - 10 = 2; 12 - 8 = 4; 12 - 6 = 6.
4. Multiply the obtained numbers (including the *medietas*) with each other: 2 x 4 x 6 x 12 = 576.
5. The square root of the obtained number will be equal to the area of the proposed triangle: $\sqrt{576} = 24$.

In the following four pages of his commentary of I.34, Scheubel applies the same rule to a multitude of examples (9 in total, counting the above) which display different types of numbers (integers, fractions, irrational numbers…). For instance, the example he considers immediately after involves irrational numbers, as here, the square root of 180.

He also teaches early on a method to perform the computation in a quicker manner (Fig. 11).

Abbreviation of the rule by economy. Since the numbers of the third multiplication, which naturally come from the first and the second multiplication, are equal to each other, which often occurs, as is evident from these two examples, the third multiplication is left aside, and also the extraction of the square root will not be necessary. But the area

of the triangle will be revealed immediately by either of two products, the first or the second.
The first product is 18, just as the second, whereby the third is 324, whose root is afterwards 18, that is, the area of the triangle, and also the half of the parallelogram of the first figure, which is what is to be shown by this rule. From the economy taught above, the third multiplication could be left aside, and the question could be immediately answered by 18 or 18, that is, by the first or by the second product, which are the same.[1]

Through this arithmetical appendix, Scheubel did not offer a numerical demontration of Euclid's Prop. I.34 properly speaking, since he did not use it to directly demonstrate that the opposite sides and angles of the parallelogrammic areas are equal to one another and that the diameter bisects them. What he proposed instead is a method useful when dealing with parallelogrammic areas in general, as they can be considered as composed of two triangles, showing thereby that Prop. I.34 can be useful to know how to determine the areas of triangles numerically. He proved thereby that, although this proposition was only applied to magnitudes by Euclid (as he noted in the introduction of his appendix), it is also true for numbers, just as the many geometrical propositions for which he applied an arithmetical and algebraic interpretation in the appendix.[2] By doing so, Scheubel implicitly established the correspondence between the generation of parallelograms and the generation of numbers by multiplication, which was often used in sixteenth-century com-

[1] Scheubel 1550, 111: "*Abbreviatio canonis per compendium.* Cum tertiæ multiplicationis numeri, qui nimirum ex prima & secunda multiplicatione proveniunt, inter se fuerint æquales, id quod sæpe contingit, in his item duobus exemplis evidens est, eadem tertia multiplicatio negligitur, nec etiam extractione radicis quadratæ tum opus erit. Verum statim per alterutrum productorum, primum vel secundum, trianguli area indicabitur. Primum productum sunt 18, secundum tantundem, tertium deinde 324. huius postea radix 18, area est trianguli, atque medietas etiam parallelogrammi vel figurae primæ, quod hoc canone ostendere oportuit. Potuisset ex compendio iam præscripto, tertia multiplicatio negligi, ac statim per 18 vel 18, primum scilicet vel secundum productum, quæstioni responderi, quod idem fuisset".

[2] Scheubel 1550, I.34, 110: "Quoniam autem *haec propositio 34, & multae etiam sequentes, in numeris, quantitate nimirum discreta, non minus atque in quantitate continua, veræ esse reperiuntur*, quo id ostendamus commodius, canonem quendam generalem, per quem omnis generis triangulorum (modo latera eorum nota fuerint) areæ inveniri possent, subijcere necesse fuit, his verbis" (my emphasis).

mentaries on Euclid to explain Euclid's Df. II.1, that is: "Any rectangular parallelogram is said to be contained by the two straight lines containing the right angle".[1]

Hence, the practical character of Scheubel's treatment of this proposition does not only come down to the numerical treatment of magnitudes, and to the fact that it offers a means to measure areas (which, as said, was a central aspect of the Latin tradition of practical geometry), but it is also related to the style of the teaching provided in this framework. Indeed, rather than merely teaching the procedure in a general and demonstrative manner, referring to undetermined or abstract lengths and areas and exhibiting the principles or prior propositions on which this procedure rests, Scheubel explained it through a series of examples that appeal to specific numerical values, in the manner of professionally-oriented mathematical manuals (such as abbacus treatises). Most of the commentary that is proposed with this set of examples provides a set of instructions on how to perform the procedure in general and as applied to the values given in the appended examples. The focus is thus set on the actions involved in the taught computational rule or algorithm and demonstrating its truth through its repeated application to different particular cases. The demonstration of the rule is also somewhat empirical since it is its application to various sets of specific numerical values (representing the lengths of the sides of various hypothetical triangles) that would allow the reader to obtain a first-hand experience of the truth, or at least of the efficiency, of the taught rule, and indirectly of Euclid's proposition.

2.5. References to artisanal applications

As mentioned earlier, the teaching of concrete applications of geometry (including of certain Euclidean propositions) was frequent in practical geometry treatises, which often taught measurement and computational practices useful

[1] Yet, interestingly, Scheubel did not comment on this when dealing with the definitions of Book II, though he added an arithmetical and algebraic treatment of the propositions contained in this book, when it was possible to do so.

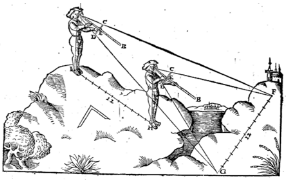
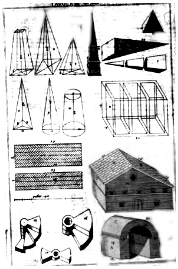
a)  b)  c) 

Figure 12: *a*) Oronce Fine, *Geometria practica*, 1532; *b*) Giovanni Pomodoro, *Geometria pratica*, 1599; *c*) Jean de Merliers, *Practique de geometrie*, 1575.

to surveyors, as well as instrument-making and mapping techniques, among other applications (Figs. 12, *a*-*c*).[1]

As said, the original text of Euclid's *Elements*, as well as most of its premodern versions, were devoid of references to exterior applications of geometry. However, in the sixteenth-century Euclidean tradition, references to artisanal uses of Euclidean concepts and propositions appear in Tartaglia's commentary on the definitions of Book XI, for the production of material artefacts or commercial arithmetic,[2] in John Dee's annotations to Billingsley's commentary on Book

[1] See, for example, Digges, *Pantometria* 1571: "The other parte named *Longimetra* the ingeniouse practitioner wil apply to *Topographie*, fortification, conducting of mines under the earth, and shooting of great ordinance". See also Peletier 1573, Problem 15, 29: "Ce que *les ouvriers* pourront faire aisément & succinctement, pour peu qu'ils ayent pratiqué la Geometrie".; Problem 17, 32: "*l'office du mesureur*, sera de chercher tele ligne perpendiculaire par artifice d'instrument"; Problem 23, 42: "La *commune usance des Artisans* est un peu plus compendieuse, mais toutesfois tiree de cete cy" (my emphasis).

[2] Tartaglia 1543, Df. XI.10, 178v: "questa diffinitione ha insegnato alli artifici il modo di formare le palle di pietra, o d'altra materea, & che'l sia il vero el si sa che se uno artifice vol fare una palla di pietra che sia perfettamente, al senso tonda lui forma prima un mezzo cerchio vacuo in qualche banda di ferro, over di legno, over daltra materia grando, over piccolo secondo la qualita della palla, over palle che desidera formare, puoi va scarpellando attorno attorno secondo l'ordine del detto vacuo di mezzo cerchio cioe giustando spesso quella forma secondo che va scarpellando & cosi pian piano la redusse a perfettione". See also Df. I.2, 6v: "Hor poniamo che sieno due misure materiale di alcuno metallo, over di legno (si come sono quelle che usano questi mechanici per misurar le cose occorrente) & che dette misure siano di egual longhezza, come serebbe che fussino duoi passi, & che

XII, for engineering and instrument-making,[1] in the 1564 French translation by the Royal lecturer Pierre Forcadel for architecture and commercial arithmetic,[2] and in Xylander and Scheubel, for surveying and commercial arithmetic.[3] I will nevertheless only consider here Forcadel's case, as it is the most exemplary case of all with regard to this issue.

In his commentary on Prop. I.43, which states that "In any parallelogram the complements of the parallelograms about the diameter are equal to one another",[4] Forcadel taught a method to determine the length of a side of a parallelogram from the knowledge of its area and of its other side which, he claimed, is used in the trade of cloth to determine the length of a piece of cloth of given width that is necessary to have in order to obtain a surface equal to that of another piece of cloth with a different length.

Let us take from here that, from two equal and rectangular complements, knowing the content of one [complement] and of one of the sides of the other, we will obtain the

ciascuno di essi passi sia diviso in cinque piedi, liquali piedi siano di onze xii come si costuma fra li Architetti".

[1] Dee, in Billingsley 1570, XII.1, 357v: "The great Mechanicall use (besides Mathematicall considerations) which, these two Corollaryes may have in *Wheeles of Milles, Clockes, Cranes, and other engines for water workes, and for warres*, and many other purposes, *the earnest and wittie Mechanicien* will soone boult out, & gladly practise" (my emphasis).

[2] Forcadel 1564, VI.4, 161v: "Et de cecy *les massons* prennent la reigle pour mesurer le plan dessus du nombril d'un puis telle que s'ensuit" ; II.4, 45r-v: "de là vient que *ceux qui traffiquent les monnoyes*, multiplient les deniers d'aloy particuliers par les marcs particuliers un chacun par le lien, & divisent les produicts adjouster ensemble, par tous les marcs proposez, & en vient les deniers de l'aligation". See also *infra*, p 84 fn.

[3] Scheubel 1550, 22-23: "Est aedificium quoddam παραλλήλως secundum quatuor eius latera extructum, cuius altitudo cum ad suam longitudinem Superbipartientem tertias, ad latitudinem vero, Duplam sesquialteram constituat rationem, altitudine deinde cum longitudine, ac producto tandem cum latitudine multiplicato, numerus 39930 ulnarum producatur, quantae huius aedificij singulae dimensiones fuerint, quaeritur (...) Murus, cuius longitudo quidem in 3½ ad latitudinem, altitudo verò in quicupla ratione ad longitudinem constructus est, ab Artifice tandem 980 coronatis redimitur. Quoniam autem, cum pro singulis virgis, ut dicitur, extruendis, tot coronati, quot ipse murus in latitudine virgas habet, expositi sint, quae nam huius muri altitudo sit, longitudo item, ac latitudo etiam, quaeritur". See also *supra* and Xylander II.1, 46: "Was grössen nuß diser vortail *in der rechenkhunst* hab, mag niemandt gnugsam erzelen. Dan es ist offenbar das die welsch practic (welche allain ist ain gschwinder, schöner khunstlicher und artlicher vortail und proceß der regel de tri) so alle andere rechnungen, die Coß ausgenomen" (my emphasis).

[4] Euclid, VI.14 (Heath 1956, I, 340).

other side. For by dividing the given content by the known side, will result the unknown content. Those who are engaged in the trading of goods (*ceux qui traffiquent la marchandise*) are used to ask this in this manner: I have bought twenty-four ells of cloth of three quarters in width. I want to double them with another type of cloth which measures two thirds in width. I ask how much of it I should take. They multiply three quarters by twenty-four, or they take the three quarters of twenty-four, which amounts to eighteen for the content of one and of the other complements, which number, divided by two thirds, amounts to twenty-seven for the unknown side of the other complement, and therefore, it will be necessary to take twenty-seven ells of the cloth that measures two thirds in width. Common people (*le vulgaire*) calls this way of proceeding (*façon de faire*) "inversed rule of three" (*reigle de trois rebource*), but we, according to Euclid in the proposition 14 of the sixth book, can name it "rule of three reciprocals" (*reigle de trois reciproques*).[1]

The method was first designated here by the name given to it by common people ("le vulgaire"), namely "reigle de trois rebource", which corresponds to the designation of this procedure in French practical arithmetic treatises,[2] but is afterwards given a more scientific name, based on Euclid's Prop. VI.14,[3] namely "reigle de trois reciproques". The problem and the method were here formulated rhetorically, using only words instead of numbers or symbols or instead of an appended computation. It is even deprived of a diagram. However, Forcadel also

[1] Forcadel 1564, I.43, 35v-36r: "Prenons d'icy que de deux suplemens esgaux & rectangles, sachant le contenu de l'un, & l'un des costez de l'autre, on aura l'autre costé: car en divisant le contenu donné par le costé congneu, il en viendra le costé incogneu: ceux qui traffiquent la marchandise, ont accoustumé de le demander ainsi, j'ay acheté vingt-quatre aulnes de drap de trois quartiers en largeur, je les veux doubler d'un autre sorte de drap qui à deux tiers en largeur, je demande combien il m'en faudra prendre. Ils multiplient trois quartiers par vingt-quatre, ou prennent les trois quarts de vingt-quatre, il en vient dix huict pour le contenu de l'un & de l'autre suplement, lequel nombre divisé par deux tiers, faict vingt-sept, pour le costé incongneu de l'autre suplement, & par ainsi, il faudra prendre vingt-sept aulnes du drap ayant deux tiers de large: le vulgaire nomme une telle façon de faire la reigle de trois rebource, mais nous avec Euclide en la 14 proposition du sixiesme livre, la pourrons nommer la reigle de trois reciproque".

[2] See, for example, Peletier's *Arithmetique* (Peletier 1554, 68-69): "*De la Regle de Troes Reverse ou Rebourse. Chapitre IX.* La Regle de Troes Rebourse s'appelle einsi: parce qu'an elle l'operacion ét au rebours de celle qui se fèt an la Regle de Troes Directe".

[3] Euclid VI.14 (Heath 1956, II, 216): "In equal and equiangular parallelograms the sides about the equal angles are reciprocally proportional; and equiangular parallelograms in which the sides about the equal angles are reciprocally proportional are equal.

always made sure to refer to the concrete objects that are considered (pieces of cloth), to the specific quantities taken as examples (24; 18; ¾; ⅔…) and to the specific units in which they are measured and counted (ells or, in French, "aulnes").[1]

Another proposition about which Forcadel referred to the concrete applications of Euclid's geometry is Prop. III.31,[2] where he mentioned the use architects make of the first part of the proposition to determine, by the means of a set-square, the uniformity of the concave parts of columns and to construct and measure rectangular parallelepipeds, which would then correspond to the shape of buildings and of their inner divisions.

From the first part of this proposition, Architects have taken the composition of the set-square, by the means of whose right angle they see very well if the flutings of the columns are correctly made, which is when the right angle of the set-square touches all sides of the concavity of each excavation (*canal*). Also architect artisans (*architectes artisans*) use the same set-square to constitute (*soustenir*) as much as to square (*esquarrir*) their rectangular parallelepipeds.[3]

The actual use of the set-square to verify the configuration of columns is only outlined in a rough manner, but gives a sense of the way in which the architect would use the instrument on site. Interestingly, Forcadel seems to distinguish here the architect properly speaking, to whom he attributed the invention of the set-square and the task of verifying the uniformity of columns in already constructed edifices, and the "artisan architect", whose task is to conduct constructions and to measure surfaces and volumes of buildings and rooms rather than to observe the formal perfection of ornemental details. Through these pre-

[1] Although this form is improper to the style of Forcadel's text, I add here its symbolic expression for an easier comprehension: $24 \cdot \frac{3}{4} = x \cdot \frac{2}{3}$; $18 = x \cdot \frac{2}{3}$; $18 : \frac{2}{3} = x$; $x = 27$.

[2] Euclid, III.1 (Heath 1956, II, 61): "In a circle the angle in the semicircle is right, that in a greater segment less than a right angle, and that in a less segment greater than a right angle; and further the angle of the greater segment is greater than a right angle, and the angle of the less segment less than a right angle".

[3] Forcadel 1564, III.1, 102r-v: "De la premiere partie de ceste proposition, les Architectes ont pris la composition de l'esquierre, par l'angle droict duquel ils voyent tres bien si les caneleures des colomnes sont justement faictes, qui est quant l'angle droict de l'esquierre touche par toute la concavité d'un chacun canal: aussi les architectes artisans se servent du mesme esquierre, tant à soustenir qu'à esquarrir leurs parallelepipedes rectangles".

cisions, Forcadel seems intent to display his allegedly broad knowledge of the activities of architects.

References to other applications (in architecture, banking, military logistics, but also astronomy and gnomonics) are also made by Forcadel in various other propositions and with varying levels of details.[1]

3. Conclusion

As was shown at the beginning of the present analysis, in the medieval and Renaissance texts that proposed a discourse on the division of geometry, theoretical and practical geometry were clearly distinguished. In this context, practical geometry was generally presented as an art of measuring specific or concrete magnitudes, or more rarely (as in Gundisalvi's *De divisione philosophiae*) as the geometrical knowledge necessary to artisans. Theoretical geometry was then described as a purely contemplative and demonstrative knowledge of the principles and properties of abstract magnitudes and was directly or indirectly identified with the geometry dealt with in Euclid's *Elements*.

Considering the treatises which were labelled as belonging to practical geometry in the Latin middle ages and in the sixteenth century, we also saw that, in spite of a number of aspects which allows us to recognise a given geometrical

[1] Forcadel 1564, II.1, 44v: "Nous prenons aussi icy la façon de *changer la valeur de quelque piece de monnoye que ce soit*, en plusieurs sortes de plus petites monnoyes, pour en avoir autant de l'une que de l'autre, comme par exemple: si quelqu'un me dict qu'il desire changer une piece de monnoye qui vaut 34 sols, en sols, en doubles, & en liards, & qu'il desire avoir autant de l'une sorte que de l'autre, alors (…)" ; Forcadel 1564, II.4, 49v: "Par ceste proposition si on nous dict, qu'*un maistre de camp, a un certain nombre de soldats, lesquels il desire metre en bataille* en telle sorte qu'ils soyent mis en une figure de quatre costez, & qu'il y en aye autant d'un costé que d'autre (…)"; Forcadel 1564, III.1, 78r: "Prenons icy que *si lon prent un gnomon ou aiguille d'arain propre à monstrer les umbres*, comme le met Vitruve au premier livre de son Architecture, & lon la met perpendiculairement sur le plan de quelque Orizon, ou sur un plan parallele, à l'un des Orizons principaux d'un chacun lieu (…)"; Forcadel 1564, V.4, 132v: "Par ceste proposition icy si on me dit que trois aulnes ½ de quelque marchandise coustent 5 livres ½, & on me demande *combien cousteront 7 aulnes de la mesme marchandise*, alors (…)" ; VI.5, 155r: "Quand lon nous dit que quant 10 hommes en 12 jours ont gaigné 15 escus, *combien gaigneront 6 hommes en 18 jours*, nous avons accoustumé de multiplier les trois derniers nombres (…)".

work as a treatise of practical geometry, there already was no clear-cut distinction between what is properly practical and what is properly theoretical with regard to their content and style, offering a theoretical treatment of a professional type of geometrical knowledge and constituting as such a hybrid between a utilitarian form of geometrical knowledge and a scholarly and speculative teaching of geometry. And throughout its early modern development, notably in the sixteenth century, the practical geometry tradition, as it was gaining in diversity of content and style, interacted in a clearer and stronger manner with the scholarly tradition of geometry represented by Euclid's *Elements*. In turn, translations and commentaries on the *Elements* increasingly took on features most often found in practical geometry treatises and which were considered as different from, or even contrary to, the essential characteristics that were typically associated with Euclid's geometry. In the second part of this article, I gave several examples to illustrate this introduction of a practical approach to geometry within the sixteenth-century Euclidean printed tradition, which go from the replacement of Euclid's rational and demonstrative proofs by more practical or empirical explanations to the fact of presenting utilitarian applications of Euclid's propositions in the commentary.

All of the considered practical features were certainly only present in a limited number within the sixteenth-century Euclidean corpus and, when some of these features were significantly common, they were not present to the same extent in all the concerned works. Also, their acknowledgement as features of a practical nature, if not self-evident or made clear by the author, mostly depended on the general intention behind the reworking of Euclid's text, on the number of different types of practical changes used by each of them, as well as on the intensity with which they were introduced.[1] Furthermore, in the works considered here, which hold an exemplary status in this respect within the sixteenth-century Euclidean tradition, certain practical elements only occupied a restricted place. Yet, they were most often introduced in a strategic place, that is, within the first book or the first propositions, through which

[1] It was unfortunately not possible in the frame of this article to provide an outline of the distribution of practical features throughout the whole corpus of sixteenth-century Euclidean tradition and within the few commentaries that were analysed here. This will be the topic of a forthcoming study.

the student would be made familiar with the fundamental concepts and modes of operation inherent to Euclid's geometry. Moreover, in comparison with the Latin medieval tradition, such practical treatment of Euclid's propositions was much more extensive and it was practical in a much stronger and more explicit manner.

The motivations behind this practical treatment of Euclid's propositions were not always clearly formulated in the context of the considered commentaries, apart to some extent in Tartaglia and Xylander. But the institutional status of the authors of these translations and commentaries of Euclid's *Elements*,[1] the type of audience to which they were primarily addressed[2] and to a certain extent the language in which they were written,[3] among other factors, allows us to consider that their intention was above all to teach Euclid's propositions in a more accessible, meaningful, useful and/or recreative manner, helping beginners to gain an appropriate understanding and memory of the geometrical content of the *Elements* in the most effective manner, guiding their use of the instruments that could be used in and out of the classroom to perform Euclid's constructions and offering them an insight of the profit that could be gain from the study of geometry.

The fact of adapting Euclid's text so that it could be used as an introduction to

[1] Nearly all were professors of mathematics. An exception would be Billingsley, who is not known to have taught mathematics.

[2] These would have mainly been their students, although the printed format allowed for a wider circulation. In the cases of Tartaglia, Xylander and Billingsley, and perhaps also Forcadel, the fact of publishing a vernacular translation of Euclid would to a certain extent allow for, and be intended to, a more diverse audience.

[3] That is, in the case of the vernacular translations of Tartaglia, Xylander, Forcadel and Billingsley. It is however important to bear in mind that the dichotomy between Latin and vernacular cannot simply be equated with the dichotomy between scholarly and lay, or between scholars and craftsmen. Indeed, as expressed by Giacomotto-Charra and Silvi (2014, 14): "La vernacularisation n'implique pas nécessairement la vulgarisation, de même que la vulgarisation ne saurais se limiter à la vernacularisation". In other words, the fact of writing a treatise on a topic generally transmitted in Latin (i.e. a language typically used by a learned audience in the sixteenth century) in the vernacular (i.e. the language of everyday life, understood by a greater number of people), or of translating a work initially written in Latin into a vernacular language, does not necessarily imply that the work was primarily addressed to a less educated or lay audience, even if it would *de facto* be more accessible to the common man, at the same time as to more erudite readers. On this issue, see also Beaujouan 1975.

practical geometry, and to practical mathematical texts more generally, is not excluded in certain cases, in particular in the work of Tartaglia or in that of Clavius, who gave a significant place to practical mathematics in the mathematical curriculum of the Jesuit colleges[1] and who wrote many practical mathematical treatises, including his 1604 *Geometria practica*. This work not only contained many references to Euclid's *Elements*, but also took up a part of its content,[2] among which his construction of the *quadratrix*.[3]

The more epistemological motivations relating to the demonstration of the mechanical origin of certain geometrical constructions may only be indirectly sensed in the commentary of Clavius, and perhaps also in that of Billingsley, who both referred to the homology between the abstractly represented generation of geometrical objects from a point, a line or a surface and the instrumentally-produced geometrical magnitudes.[4] However, such a discourse was not then related to the practical interpretation of Euclid's propositions that was offered by these authors, being rather set forth in the context of a more universal discourse on the essential properties of geometrical objects. Also, in Billingsley's case, such a correlation between the mechanical and abstract generations of figures may have had a primarily pedagogical purpose, aiming to help the reader better grasp the meaning of Euclid's definitions.

Also, it is yet unclear to me whether these authors consciously or unconsciously aimed to contribute thereby to the constitution of a new approach to geometry that would incorporate or conjoin both approaches to geometry, theoretical and practical, into one comprehensive type of geometrical teaching, or even that they conceived this to be possible or desirable.[5] Yet, by proposing a more extensive and explicit practical treatment of parts of Euclid's *Elements*, they offered *de facto* to their readers, who were mainly students and profes-

[1] On Clavius and the Jesuit *ratio studiorum*, see Romano 1999, chap. 2-3, Smolarski 2002, and Gatto 2006.

[2] Knobloch 1997.

[3] On Clavius' construction of the quadratrix, a curve intended to solve the quadrature of the circle, in his commentary on Euclid and in his *Geometria practica*, see Bos 2001, pp. 160-166, and Axworthy 2022, 217-219.

[4] Axworthy 2022, 146-147, 157 and 199-202.

[5] A promising case in this regard may be found in the approach of Joachim Jungius in the seventeenth century, as shown by Friedman 2022.

sors (though they would also be of interest to other types of readers, such as erudite notables, members of the administration and learned artisans) a new representation of Euclidean geometry in which the objects, methods and argumentative style of scholarly geometry mingle with those of a more practical and useful type of geometrical knowledge. As such, translations and commentaries of Euclid such as those I presented here effectively contributed to erode the distinction between theoretical and practical geometry that circulated in pre- and early modern Western mathematical culture, concretely demonstrating the mutual dependence of each of the two subdivisions of geometry on the other. This certainly does not mean that works pertaining more strictly to one or the other kind of geometrical teaching did not continue to be written and published up to the nineteenth century (at least), but the practical reworking of Euclid made by commentators such as Scheubel, Forcadel or Clavius provided, besides such works, the representation of a properly hybrid approach to geometry through which the theoretical is made practical.

It is important to note, furthermore, that just as there was not one type of approach to practical geometry in the pre- and early modern era, there was not one type of practical approach to Euclid's *Elements* adopted by its commentators, one (as Scheubel) focusing mostly on the numerical treatment of Euclid's geometrical propositions, another (as Clavius) on the purely constructive versions of Euclid's problems, and another yet (such as Forcadel) on their utilitarian applications. The way Euclid was made practical in the sixteenth century thus offers a representation of the multitude of different ways in which could be defined practical knowledge in the field of geometry. Therefore, the practical approach to Euclid's principles and propositions in the sixteenth-century printed tradition did not only contribute to changing the image, content and style of theoretical geometry, as represented by Euclid's *Elements*, but also the very image of practical geometry, which was, in this context, less and less restricted to an art of measuring and representing more generally any form of hands-on, socially-relevant, useful and recreative geometry.

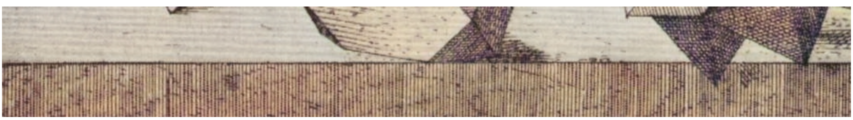


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Table 1. Zamberti, in Lefèvre d'Étaples, *Geometricorum elementorum libri XV*, 1516, I.11, 9r, Tours, Centre d'Études Supérieures de la Renaissance; Scheubel, *Euclidis Megarensis*, 1550, I.3, p. 84, München, Bayerische Staatsbibliothek.

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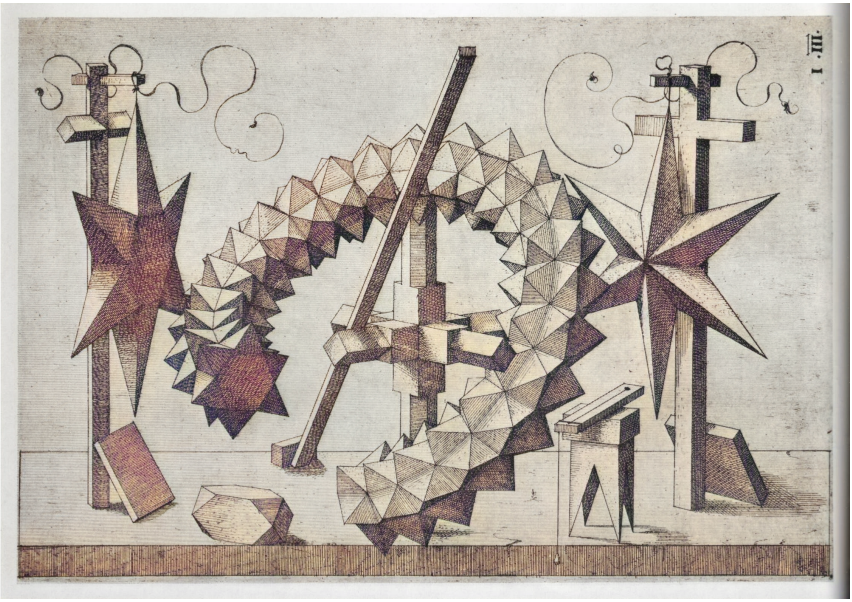

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